Chapter 2 Basics of statements, logic

Today we'll talk more about statements and equivalences.

DeMorgan's Laws

1. $\neg(P \vee Q) \Longleftrightarrow \neg P \wedge \neg Q$
2. $\neg(P \wedge Q) \Longleftrightarrow \neg P \vee \neg Q$

Negation of implication $\neg(P \Rightarrow Q) \Longleftrightarrow P \wedge \neg Q$

Example Make truth table for $\neg(P \Rightarrow Q)$ and for $P \wedge \neg Q$

P	Q	$P \Rightarrow Q$	$\neg(P \Rightarrow Q)$	$\neg Q$	$P \wedge \neg Q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

Since the columns for $\neg(P \Rightarrow Q)$ and $P \wedge \neg Q$ are the same they are logically equivalent.

Example Negate the implications

(a) "if f is a continuous function then f is differentiable"

(b) "if x is a real number then x is even or odd"

(a) "f is a continuous function and not differentiable"

(b) "x is a real number and it is neither even nor odd"

Definition A statement whose truth values are all T is called a tautology. One whose truth values are all F is called a contradiction.

Example Construct truth tables for $P \vee \neg P$ and $P \wedge \neg P$.

P	$\neg P$	$P \vee \neg P$	$P \wedge \neg P$
T	F	T	F
F	T	T	F

$P \vee \neg P$ is always true

"I like pizza or I don't like pizza" tautology!

$P \wedge \neg P$ is always false contradiction!

Example (practice with connecting 3 statements) Make a truth table for $\neg(P \wedge Q) \vee \neg R$

P	Q	R	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg R$	$\neg(P \wedge Q) \vee \neg R$
T	T	T	T	F	F	F
T	F	T	F	T	F	T
F	T	T	F	T	F	T
F	F	T	F	T	F	T
T	T	F	T	F	T	T
T	F	F	F	T	T	T
F	T	F	F	T	T	T
F	F	F	F	T	T	T

You might have been able to guess this would happen by doing some algebra:

$$\neg(P \wedge Q) \vee \neg R \iff \neg((P \wedge Q) \wedge R)$$

$$\iff \neg(\underbrace{P \wedge Q \wedge R}_{\text{only T when } P, Q, R \text{ are all T}})$$

Algebraic properties

(1) $\neg(P \vee Q) \Longleftrightarrow \neg P \wedge \neg Q$

(2) $\neg(P \wedge Q) \Longleftrightarrow \neg P \vee \neg Q$

$\Big\}$ De Morgan's laws

(3) $P \vee Q \Longleftrightarrow Q \vee P$

(4) $P \wedge Q \Longleftrightarrow Q \wedge P$

$\Big\}$ commutativity

(5) $P \wedge (Q \wedge R) \Longleftrightarrow (P \wedge Q) \wedge R$

(6) $P \vee (Q \vee R) \Longleftrightarrow (P \vee Q) \vee R$

$\Big\}$ associativity

(7) $P \wedge (Q \vee R) \Longleftrightarrow (P \wedge Q) \vee (P \wedge R)$

(8) $P \vee (Q \wedge R) \Longleftrightarrow (P \vee Q) \wedge (P \vee R)$

$\Big\}$ distributivity

Problem 1. Consider the statements $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee (P \wedge R)$. Make a truth table for both statements and state why you can conclude they're equivalent.

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T	T	T	T	T	T
T	F	T	T	T	F	T	T
F	T	T	T	F	F	F	F
F	F	T	T	F	F	F	F
T	T	F	T	T	T	F	T
T	F	F	F	F	F	F	F
F	T	F	T	F	F	F	F
F	F	F	F	F	F	F	F

They have the same truth values, so $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee (P \wedge R)$ are equivalent.

Problem 2. Consider the following statements. Make a truth table for each. What simpler statement are they both equivalent to?

a. $P \vee (P \wedge Q)$

b. $P \wedge (P \vee Q)$

P	Q	$P \wedge Q$	$P \vee (P \wedge Q)$	$P \vee Q$	$P \wedge (P \vee Q)$
T	T	T	T	T	T
T	F	F	T	T	T
F	T	F	F	T	F
F	F	F	F	F	F

Both statements are equivalent to P. This is called the "absorption" property by some.

Problem 3. The following are examples of implications (ie. statements of the form $P \implies Q$). Rephrase them so that they're in the form "if P then Q" and state which part is the hypothesis P and which part is the conclusion Q.

a. The sum of two positive numbers is positive.

b. The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the two other sides.

c. All primes are even.

d. The square of every real number is positive.

(a) $P =$ " x and y are positive" $=$ " $x > 0$ and $y > 0$"

$Q =$ " $x+y$ is positive" $=$ " $x+y>0$"

(b) $P =$ " c is the length of the hypotenuse of a right triangle and a, b are the lengths of the other sides"

$Q =$ " $c^2 = a^2 + b^2$"

(c) $P =$ " x is a prime number"

$Q =$ " x is even"

(d) $P =$ " x is a real number"

$Q =$ " x^2 is positive"

Problem 4. Express the negation of each statement in the previous problem.

(a) $P \wedge \neg Q =$ " x and y are positive and $x+y$ is not positive " $=$ " $x>0$ and $y>0$ and $x+y \leq 0$ "

(b) $P \wedge \neg Q =$ " c is the length of the hypotenuse of a right triangle and a, b are the lengths of the other sides and $c^2 \neq a^2+b^2$ "

(c) $P \wedge \neg Q =$ " x is a prime number and x is not even" $=$ " there exists a prime number that is not even"

(d) $P \wedge \neg Q =$ " x is a real number and x^2 is not positive" $=$ " there exists a real number whose square is not positive"

Problem 5. Just for fun, discuss whether the statement is true or the negation is true for each of the statements in Problems 3 and 4.

(a) $P \Rightarrow Q$ is true

(b) $P \Rightarrow Q$ is true (Pythagorean theorem)

(c) $P \wedge \neg Q$ is true (for example 3 is a non-even prime)

(d) $P \wedge \neg Q$ is true (0, note positive means strictly greater than 0)

Problem 6. A *deductive argument* is a collection of statements $A_1, A_2, \ldots, A_n$ called *premises or hyptotheses* followed by a statement B called the *conclusion*. A deductive argument is called *valid* if whenever $A_1, A_2, \ldots, A_n$ are all true, B is true as well. In symbols, we can write a deductive argument as

$$\begin{array}{l} A_1 \\ A_2 \\ \vdots \\ A_n \\ -- \\ \therefore B \end{array}$$

To check whether a deductive argument is valid, we can make a truth table using $A_1, \ldots, A_n$ and B and check that whenever $A_1, \ldots, A_n$ are all true, B is true as well. Make a truth table using the statements in the following argument and decide whether it is valid.

$$\begin{array}{l} P \implies \neg Q \\ R \implies Q \\ R \\ -- \\ \therefore \neg P \end{array}$$

P	Q	$\neg Q$	R	$P \Rightarrow \neg Q$	$R \Rightarrow Q$	$\neg P$
T	T	F	T	F	T	F
T	F	T	T	T	F	F
F	T	F	$\boxed{T}$	$\boxed{T}$	$\boxed{T}$	$\boxed{T}$
F	F	T	T	T	F	T
T	T	F	F	F	T	F
T	F	T	F	T	T	F
F	T	F	F	T	T	T
F	F	T	F	T	T	T

The argument is valid