Chapter 14 Functions, Domain, and Range

Recall that a relation on $A \times B$ is a subset of $A \times B$.

Def A function f from A to B, denoted $f : A \to B$, is a relation on $A \times B$ such that

1. $\forall\, x \in A$, $\exists\, y \in B$ such that $(x, y) \in f$.

   "every input has an output"

2. if $(a, b) \in f$ and $(a, c) \in f$, then $b = c$.

   "every input has a unique output"

When these conditions are satisfied, we often say that the function is well-defined. The set A is called the domain of f, denoted $\mathrm{dom}(f)$, and the set B is called the codomain of f, denoted $\mathrm{cod}(f)$.

Exercise 14.1. Let $A = \{1,2,3\}$ and $B = \{2,4,6\}$. Which of the following are functions from A to B? If they are not functions, explain which rule is violated.

(a) The relation f is $\{(1,2),(2,4),(3,4)\}$.
(b) The relation f is $\{(1,2),(1,4),(2,2),(3,6)\}$.
(c) The relation f is $\{(1,2),(3,4)\}$.
(d) The relation f is $\{(2,4),(1,2),(3,6)\}$.

Exercise 14.4. Decide which of the following are functions and which are not, giving reasons for your answers.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x+2$.
(b) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1/x^2$.
(c) Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(x) = (x,x)$.
(d) Let $f : \mathbb{Q} \to \mathbb{Q}$ be defined by $f(p/q) = 1/q$, where p and q are integers and $q \neq 0$.
(e) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x,y) = (x,3)$.

Exercise 14.6. For each of the two examples below decide whether or not the object so defined is a function. Give reasons for your answers.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -(x^2) & \text{if } x \leq 0 \end{cases}.$$

(b) Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by

$$f(x) = \begin{cases} 1 \text{ if } x \in 2\mathbb{Z} \\ 2 \text{ if } x \text{ is prime} \\ 3 \text{ otherwise} \end{cases}.$$

Exercise 14.9. What is the range of each of the functions below? A picture, when appropriate, is a lovely addition and is heartily encouraged. It does not, however, substitute for the real thing. Write out everything explicitly.

(a) The function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = 1/x$.
(b) The function $f : \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \to \mathbb{R}$ defined by $f(x,y) = x/y$.
(c) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 4x + 5$.

The "codomain" is maybe terminology that is new to you.

We think of it as the set of possible output values.

This is different from the set of output values.

Def Given a function $f: A \to B$, the range of f is given by the set

$$\operatorname{ran}(f) = \{ f(a) : a \in A \}$$
$$= \{ b \in B : \exists\, a \in A \text{ such that } f(a) = b. \}.$$

Example Determine the range of $f: \mathbb{R} \setminus \{3\} \to \mathbb{R}$ given by $f(x) = \dfrac{x+1}{x-3}$ and prove your claim.

Preliminary work We need to find all y such that $y = \dfrac{x+1}{x-3}$. What if we solve this for x?

$$xy - 3y = x + 1$$
$$\Leftrightarrow x(y-1) = 3y + 1$$
$$\Leftrightarrow x = \frac{3y+1}{y-1}$$

So to get a particular output y, we can use the formula $x = \frac{3y+1}{y-1}$ for the input that gives output y. The issue it seems is the formula fails when $y=1$. This cannot be part of our range we claim

<u>Claim</u> $\operatorname{ran}(f) = \mathbb{R} \setminus \{1\}$. Let $y \in \operatorname{ran}(f)$.
Since $y \in \operatorname{ran}(f)$, there exists $x \in \operatorname{dom}(f)$ such that $\frac{x+1}{x-3} = y$. We claim $y \neq 1$. Suppose $y=1$. Then $\frac{x+1}{x-3} = 1$

and so $x+1 = x-3$, which implies $1 = -3$, a contradiction.
Thus $y \in \mathbb{R} \setminus \{1\}$, and so $\operatorname{ran}(f) \subseteq \mathbb{R} \setminus \{1\}$. Conversely, suppose $y \in \mathbb{R} \setminus \{1\}$. We claim $y \in \operatorname{ran}(f)$. We must show there exists $x \in \operatorname{dom}(f)$ such that $f(x) = y$. We claim $x = \frac{3y+1}{y-1}$ works. Observe that

$$f\left(\frac{3y+1}{y-1}\right) = \frac{\frac{3y+1}{y-1} + 1}{\frac{3y+1}{y-1} - 3}$$

$$= \frac{\frac{3y+1}{y-1} + \frac{y-1}{y-1}}{\frac{3y+1}{y-1} - \frac{3y-3}{y-1}}$$

$$= \frac{3y+1+y-1}{3y+1-(3y-3)} = \frac{4y}{4} = y.$$

Thus $\mathbb{R} \setminus \{1\} \subseteq \operatorname{ran}(f)$ and so $\operatorname{ran}(f) = \mathbb{R} \setminus \{1\}$.