Chapter 15 Functions, one-to-one, and onto

Def A function $f: A \to B$ is said to be **one-to-one** (or **injective**) if for all $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$, then $a_1 = a_2$. Every value in the range has a unique corresponding value in the domain.

"Horizontal Line Test" if there exists a horizontal line that passes through 2 points of the graph of f, then f is not 1-1.

The function is said to be **onto** (or **surjective**) if $\operatorname{ran}(f) = B$ (i.e. for all $b \in B$, there exists $a \in A$ such that $f(a) = b$.).

If a function is both 1-1 and onto then we say that it is **bijective**.

Example The function $f: \mathbb{R} \setminus \{3\} \to \mathbb{R}$ given by $f(x) = \dfrac{x+1}{x-3}$ is 1-1 but not onto.

Proof Suppose $x_1, x_2 \in \mathbb{R} \setminus \{3\}$ and $f(x_1) = f(x_2)$. Then $\dfrac{x_1+1}{x_1-3} = \dfrac{x_2+1}{x_2-3}$ which implies $(x_1+1)(x_2-3) = (x_2+1)(x_1-3)$.

Multiplying out both sides and further simplifying gives $x_1 = x_2$

Thus f is 1-1. On the other hand, we've shown that $\operatorname{ran}(f) = \mathbb{R} \setminus \{1\}$. Thus $\operatorname{ran}(f) \neq \operatorname{cod}(f)$, so f is not onto.

Example Show that $f : \mathbb{Z} \to \mathbb{N}$ given by

$$f(n) = \begin{cases} 2n & n \geq 0 \\ -2n-1 & n < 0 \end{cases}$$

- non-negative integers get mapped to even naturals
- negative integers get mapped to odd naturals.

is a bijection.

Proof We first show f is onto. Let $m \in \mathbb{N}$. We must show there exists $n \in \mathbb{Z}$ such that $f(n) = m$. We consider two cases: m is even or m is odd. Suppose m is even. Then $m = 2k$ for some $k \in \mathbb{N}$. Let $n = k = \frac{m}{2}$. Then $f(n) = f(k) = 2k = m$. Suppose m is odd. Then $m = 2k-1$ for some $k \in \mathbb{N}$. Let $n = -k$. Then $f(n) = f(-k) = -2(-k) - 1 = 2k - 1 = m$.

Next we show f is 1-1. Suppose $n_1, n_2 \in \mathbb{Z}$ and $f(n_1) = f(n_2)$. We consider 3 cases: (1) n_1 and n_2 both non-negative, (2) n_1 and n_2 both negative, and (3) one negative and one non-negative.

In the first case, $f(n_1) = f(n_2)$ means $2n_1 = 2n_2$ and so $n_1 = n_2$. In the second case, $f(n_1) = f(n_2)$ means $-(2n_1 + 1) = -(2n_2 + 1)$ and by cancelling, again we have $n_1 = n_2$. Finally, in the last, suppose without loss of generality $n_1 < 0 \leq n_2$. Then $f(n_1) = f(n_2)$ means $-2n_1 - 1 = 2n_2$. But this is a contradiction. the left side is odd and the right side is even.

Thus in the valid cases, $n_1 = n_2$ and f is one-to-one.

Problem 1. For each of the following functions, determine whether it is one-to-one or onto or both. Explain your reasoning.

a. $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $f(x) = 1/x$

b. $f : \mathbb{Z} \times (\mathbb{Z} \times \setminus \{0\}) \to \mathbb{R}$ given by $f(x, y) = x/y$

c. $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 + 4x + 5 = (x + 2)^2 + 1$

(a) The function is 1-1. Let $x_1, x_2 \in \mathbb{R}\setminus\{0\}$. Suppose $f(x_1) = f(x_2)$. Then $\frac{1}{x_1} = \frac{1}{x_2}$, which implies by algebra that $x_1 = x_2$. The function is not onto. That is, we claim $\exists\ y \in \mathbb{R}$ such that $f(x) \neq y$ for all $x \in \mathbb{R}\setminus\{0\}$. Indeed consider $y = 0$. Observe that for any $x \in \mathbb{R}\setminus\{0\}$, $\frac{1}{x} \neq 0$.

(b) The function is not 1-1. That is, there exists $(x_1, y_1), (x_2, y_2) \in \mathbb{Z} \times (\mathbb{Z}\setminus\{0\})$ such that $f(x_1, y_1) = f(x_2, y_2)$ but $(x_1, y_1) \neq (x_2, y_2)$. Indeed, consider $(x_1, y_1) = (2, 1)$ and $(x_2, y_2) = (4, 2)$. Then $f(x_1, y_1) = 2 = f(x_2, y_2)$ but $(2, 1) \neq (4, 2)$. The function is not onto. We claim $\exists\ z \in \mathbb{R}$ such that $f(x, y) \neq z$ for all $(x, y) \in \mathbb{Z} \times (\mathbb{Z}\setminus\{0\})$. Indeed, let $z = \sqrt{2}$. We claim $f(x, y) \neq \sqrt{2}$ for all

$(x,y) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. Suppose not then $\exists$ $x, y \in \mathbb{Z}, y \neq 0$ such that $\sqrt{2} = \frac{x}{y}$. But this is a contradiction since it implies $\sqrt{2} \in \mathbb{Q}$.

ⓒ The function is not 1-1. That is we claim

$\exists$ $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Indeed, consider $x_1 = 0$ and $x_2 = -4$. Then

$f(x_1) = 5 = f(x_2)$ but $x_1 \neq x_2$.

The function is not onto. That is, we claim

$\exists$ $y \in \mathbb{R}$ such that $f(x) \neq y$ for all $x \in \mathbb{R}$.

Indeed, let $y = 0$ (or any $y < 1$ would work).

Then, since $f(x) \geq 1$ for all $x \in \mathbb{R}$, $f(x) \neq y$.