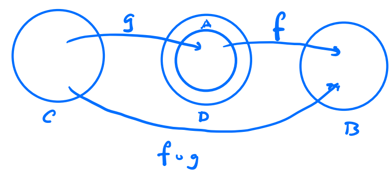
Chapter 16 Inverses

Def Let $f: A \to B$ and $g: C \to D$ be given functions. If $\operatorname{ran}(g) \subseteq A$, we can define the composition of f and g to be the function $f \circ g : C \to B$ given by $(f \circ g)(x) = f(g(x))$.



Order matters: $f, g: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ and $g(x) = \sin x$. Then $(f \circ g)(x) = (\sin x)^2 = \sin^2 x$.

But $(g \circ f)(x) = \sin(x^2) \neq \sin^2 x$.

Domain / codomain / range matters: Let $f: \mathbb{R} \to \mathbb{R}$ and $g: (0, \infty) \to \mathbb{R}$ be given by $f(x) = \frac{x}{2}$ and $g(x) = \ln(x)$. Then $f \circ g$ is defined (it's $(f \circ g)(x) = \frac{\ln x}{2}$) but $g \circ f$ is not unless we restrict the domain of f since we cannot plug in negative numbers into g.

Remark Function composition has the associative property: $(f \circ g) \circ h = f \circ (g \circ h)$.

Def The identity function on a set A is a function $i_A : A \to A$ given by $i_A(x) = x$

Question Which of the following functions f, g satisfy $f \circ g = i_{\mathbb{R}}$?

① $f(x) = 2x, \quad g(x) = \frac{x}{2}$

② $f(x) = x^3, \quad g(x) = x^{1/3}$

③ × $f(x) = x^2, \quad g(x) = \sqrt{x}$ doesn't work

since $f \circ g = i_{[0,\infty)}$ since $\mathrm{dom}(g) = [0,\infty) \neq \mathbb{R}$

and note $g \circ f \neq i_{\mathbb{R}}$ since $(g \circ f)(x) = |x|$

④ × $f(x) = e^x, \quad g(x) = \ln x$ doesn't work

since $f \circ g = i_{(0,\infty)}$ since $\mathrm{dom}(g) = (0,\infty) \neq \mathbb{R}$

but $g \circ f = i_{\mathbb{R}}$

Def Let $f: A \to B$ be a bijection. Then the inverse of f is a function $f^{-1}: B \to A$ such that

$$f^{-1}(y) = x \quad \text{if and only if} \quad f(x) = y.$$

f^{-1} is a well-defined function:

① f onto $\Rightarrow$ $\forall\, y \in B$ $\exists$ $x \in A$ such that $f(x) = y$

$\Rightarrow$ every input of f^{-1} has an output

② $f^{-1}(b) = a$ and $f^{-1}(b) = c$ $\Rightarrow$ $f(a) = b = f(c)$

and f one-to-one $\Rightarrow$ $a = c$

$\Rightarrow$ every input of f^{-1} has a unique output.

Theorem Let $f: A \to B$ be a bijection. Then

① $f \circ f^{-1} = i_B$

② $f^{-1} \circ f = i_A$

③ f^{-1} is a bijection

④ If $g: B \to A$ is a function such that $g \circ f = i_A$ or $f \circ g = i_B$, then $g = f^{-1}$.

Proof of ① Let $y \in B$ and let $f^{-1}(y) = x$.

Since $f^{-1}(y) = x$, we have by definition of f^{-1} that $f(x) = y$.

Therefore $(f \circ f^{-1})(y) = f(f^{-1}(y)) = f(x) = y$ which implies

$f \circ f^{-1} = i_B$.

Proof of ② Let $x \in A$ and let $y = f(x)$. By

definition of f^{-1}, we have $f^{-1}(y) = x$. Therefore

$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x$ which implies $f^{-1} \circ f = i_A$.

Proof of ③ We first show f^{-1} is one-to-one

Suppose $f^{-1}(y_1) = f^{-1}(y_2)$. Then taking f of both sides:

$$y_1 = f(f^{-1}(y_1)) = f(f^{-1}(y_2)) = y_2.$$

Next we show f^{-1} is onto. Let $a \in A$. We must

show there exists $b \in B$ such that $f^{-1}(b) = a$. Let

$b = f(a)$. Then $f^{-1}(b) = f^{-1}(f(a)) = a$.

Proof of ④ Suppose $g \circ f = i_A$. Observe that

$$f^{-1} = i_A \circ f^{-1} = (g \circ f) \circ f^{-1} = g \circ (f \circ f^{-1}) = g \circ i_B = g.$$

Suppose $f \circ g = i_B$. Then

$$f^{-1} = f^{-1} \circ i_B = f^{-1} \circ (f \circ g) = (f^{-1} \circ f) \circ g = i_A \circ g = g.$$