

## Chapter 18 More induction

Theorem (Second principle of mathematical induction,

also known as Strong Induction) For each  $n \in \mathbb{Z}^+$ ,

let  $P(n)$  denote a logical statement. Suppose that

(1)  $P(1)$  is true (base case)

(2) for each  $n \in \mathbb{Z}^+$ ,  $\underbrace{P(1), \dots, P(n)}_{\text{induction hypothesis}} \Rightarrow P(n+1)$ . (induction step)

Then  $P(n)$  is true for all  $n \in \mathbb{Z}^+$ .

Example Define a sequence  $(a_n)_{n=1}^{\infty}$  by  $a_1 = 1$ ,  $a_2 = 3$

and  $a_n = a_{n-1} + a_{n-2}$  for all  $n \geq 3$ . Prove that

$$a_n < \left(\frac{7}{4}\right)^n \text{ for all } n \in \mathbb{Z}^+.$$

Proof We proceed by strong induction. The base case ( $n=1$ )

is  $1 < \left(\frac{7}{4}\right)^1$  which is clearly true. Let  $n \in \mathbb{Z}^+$  and suppose

$a_k < \left(\frac{7}{4}\right)^k$  for all  $k = 1, \dots, n$ . Observe that

$$a_{n+1} = a_n + a_{n-1}$$

$$< \left(\frac{7}{4}\right)^n + \left(\frac{7}{4}\right)^{n-1} \quad \text{by induction hypothesis}$$

$$= \left(\frac{7}{4}\right)^n \left(1 + \frac{4}{7}\right)$$

$$= \left(\frac{7}{4}\right)^n \left(\frac{11}{7}\right)$$

$$< \left(\frac{7}{4}\right)^n \left(\frac{7}{4}\right)$$

$$= \left(\frac{7}{4}\right)^{n+1}.$$

Therefore, by strong induction, the inequality holds for all  $n \in \mathbb{Z}^+$ .

Example Consider the sequence  $(a_n)_{n=1}^{\infty}$  defined by

$$a_1 = 1, a_2 = 1, a_n = a_{n-1} + a_{n-2} - n + 4 \text{ for all } n \geq 3.$$

Prove that  $a_n \geq n$  for all  $n \geq 3$ .

Proof We proceed by strong induction. The base case

is  $a_3 \geq 3$  which is true since  $a_3 = 1 + 1 - 3 + 4 = 3$ .

Suppose  $n \in \mathbb{Z}^+$  such that  $n \geq 3$  and suppose  $a_k \geq k$

for all  $k = 3, \dots, n$ . Observe that

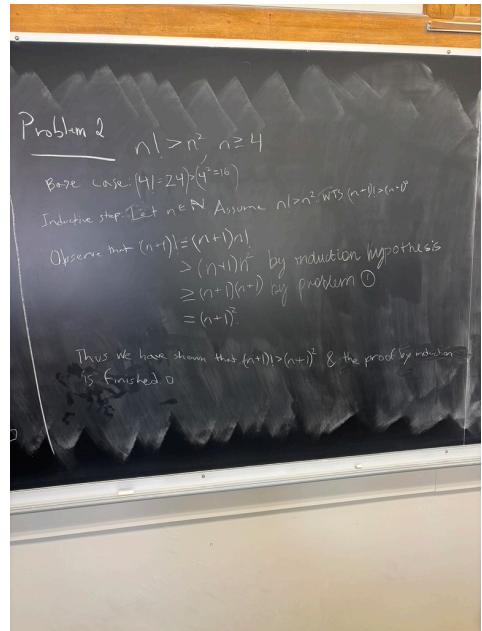
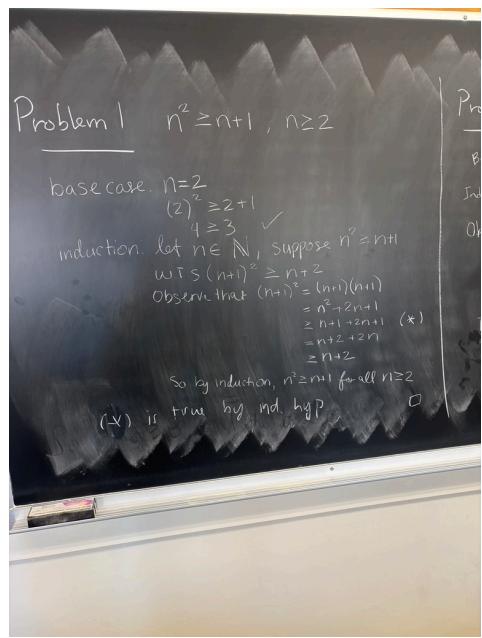
$$a_{n+1} = a_n + a_{n-1} - (n+1) + 4$$

$$\geq n + (n-1) - (n+1) + 4 \quad \text{by induction hypothesis}$$

$$= n + 2$$

$$> n + 1$$

Therefore the inequality holds for all  $n$  by strong induction.



Problem 3 Prove that  $\sum_{k=0}^n 2^k = 2^{n+1} - 1$  for any  $n \in \mathbb{Z}^+$

Proof: We will proceed by induction.

The base case  $n=0$  says that  $2^0 = 2^{0+1} - 1 = 1$ . That is clearly true.

For the induction step,

Let  $n \in \mathbb{Z}^+$  and suppose that  $\sum_{k=0}^n 2^k = 2^{n+1} - 1$

Our goal is to show that  $\sum_{k=0}^{n+1} 2^k = 2^{(n+1)+1} - 1 = 2^{n+2} - 1$

$$\begin{aligned} \text{Observe that } \sum_{k=0}^{n+1} 2^k &= \sum_{k=0}^n 2^k + 2^{n+1} \\ &= 2^{n+1} - 1 + 2^{n+1} \quad (\text{by induction hypothesis}) \\ &= 2^1 \cdot 2^{n+1} - 1 \\ &= 2^{n+2} - 1 \end{aligned}$$

By induction,  $\sum_{k=0}^n 2^k = 2^{n+1} - 1$  for any  $n \in \mathbb{Z}^+$

$$P4 a) T_1 = 1 + \sum_{k=0}^0 T_k = 2$$

$$T_2 = 1 + \sum_{k=0}^1 T_k = 1 + T_0 + T_1 = 4$$

$$T_3 = 1 + \sum_{k=0}^2 T_k = 1 + T_0 + T_1 + T_2 = 8$$

$$T_4 = 1 + \sum_{k=0}^3 T_k = 1 + T_0 + T_1 + T_2 + T_3 = 16$$

b) We will proceed by strong induction to prove  $T_n = 2^n$ .

For the base case,  $n=0$ . So  $T_0 = 1 = 2^0$ , and the equation holds.

For the induction step, let  $n \in \mathbb{N}$  and  $T_k = 2^k$  for all  $k=0, \dots, n$ .

We will show  $T_{n+1} = 2^{n+1}$  for all  $n \in \mathbb{N}$ . Observe that

$$\begin{aligned} T_{n+1} &= 1 + T_0 + T_1 + \dots + T_n \\ &= 1 + 2^0 + 2^1 + \dots + 2^n \quad \text{by induction hypothesis} \\ &= 1 + 2^{n+1} - 1 \quad \text{by Problem 3} \\ &= 2^{n+1}. \end{aligned}$$

By strong induction the equation holds for all  $n \in \mathbb{N}$ .

### Problem 5

We proceed by strong induction. For the base case,  $n=0$ ,

we have  $\frac{a^0 - b^0}{a-b} = 0 = F_0$ . For the induction step,

suppose  $n \in \mathbb{N}$  and  $F_k = \frac{a^k - b^k}{a-b}$  for all  $k=0, \dots, n$ .

Observe that

$$\begin{aligned}
F_{n+1} &= F_n + F_{n-1} \\
&= \frac{a^n - b^n}{a-b} + \frac{a^{n-1} - b^{n-1}}{a-b} \quad \text{by ind. hypothesis} \\
&= \frac{a^n + a^{n-1} - (b^n + b^{n-1})}{a-b} \\
&= \frac{a^{n-1}(a+1) - b^{n-1}(b+1)}{a-b} \\
&= \frac{a^{n-1} \cdot a^2 - b^{n-1} \cdot b^2}{a-b} \quad \text{since } a, b \text{ are} \\
&\quad \text{solutions to } x^2 = x + 1 \\
&= \frac{a^{n+1} - b^{n+1}}{a-b}.
\end{aligned}$$

Therefore by strong induction, the formula holds for all  $n \in \mathbb{N}$ .