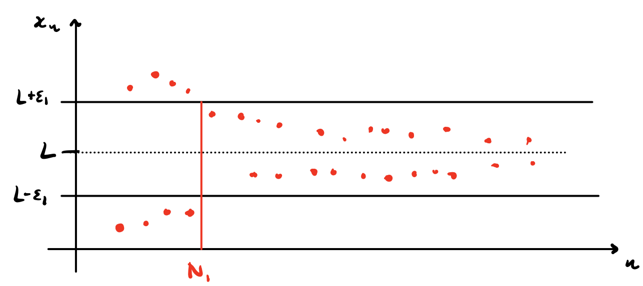
Chapter 2: Convergence of sequences

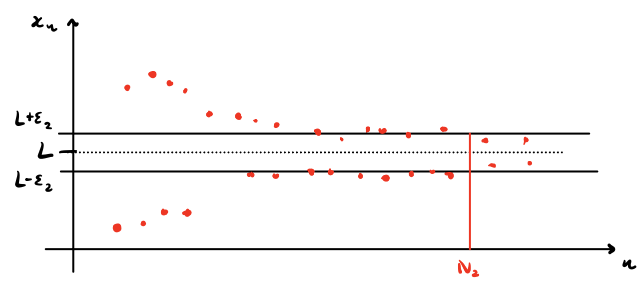
Def We say a sequence (x_n) converges if there exists $L \in \mathbb{R}$ such that for all $\varepsilon > 0$, there exists $N \in \mathbb{R}$ such that $|x_n - L| < \varepsilon$ for all $n > N$.

If such an L exists, we call L the limit of the sequence and we say (x_n) converges to L.

We write $x_n \to L$ as $n \to \infty$ or $\lim_{n \to \infty} x_n = L$.

If no such L exists, we say (x_n) diverges.





Example Let $x_n = \frac{1}{n}$ for all $n \in \mathbb{Z}$. Prove $\lim_{n\to\infty} x_n = 0$.

Proof Let $\varepsilon > 0$. Define $N = \underline{1/\varepsilon}$. Suppose $n > N$.

Then observe that

$$
\begin{aligned}
|x_n - 0| &= \left|\frac{1}{n}\right| \\
&= \frac{1}{n} \\
&< \frac{1}{N} \\
&= \frac{1}{1/\varepsilon} \\
&= \varepsilon.
\end{aligned}
$$

Example Let $x_n = \frac{n}{n+2}$ for all $n \in \mathbb{N}$. Prove $\lim_{n\to\infty} x_n = 1$.

Proof Let $\varepsilon > 0$. Define $N = \underline{\varepsilon/2}$. Suppose $n > N$.

Then observe that

$$
\begin{aligned}
|x_n - 1| &= \left|\frac{n}{n+2} - 1\right| \\
&= \left|\frac{n - (n+2)}{n+2}\right| \\
&= \left|\frac{-2}{n+2}\right| \\
&= \frac{2}{n+2}
\end{aligned}
$$

$$< \frac{2}{n}$$

$$< \frac{2}{N}$$

$$= \varepsilon.$$

<u>Example</u> Let $x_n = (-1)^n$ for all $n \in \mathbb{N}$. Prove (x_n) diverges.

<u>Proof</u> Suppose (x_n) converges. Then there exists $L \in \mathbb{R}$ such that for any $\varepsilon > 0$, there exists $N \in \mathbb{R}$ such that $|x_n - L| < \varepsilon$ for all $n > N$. Therefore if $\varepsilon = 1$ there exists $N_1 \in \mathbb{R}$ such that $|x_n - L| < \frac{1}{2}$ whenever $n > N_{1/2}$. Suppose $n > N_{1/2}$. Then

$$\begin{aligned} 2 &= |x_n - x_{n+1}| \\ &= |x_n - L + L - x_{n+1}| \\ &\le |x_n - L| + |x_{n+1} - L| \\ &< 1/2 + 1/2 \\ &= 1 \end{aligned}$$

which is a contradiction.