Chapter 20 More on convergence

Example (Warm up) Let $x_n = \dfrac{3n^2+4n+4}{5n^2+n+3}$. Find $\lim_{n\to\infty} x_n$ and prove it.

Proof We claim $\lim_{n\to\infty} x_n = \frac{3}{5} =: L$. Let $\varepsilon > 0$ and define $N = \frac{28}{25\varepsilon}$. Suppose $n > N$ and observe that

$$
\begin{aligned}
|x_n - L| &= \left| \frac{3n^2+4n+4}{5n^2+n+3} - \frac{3}{5} \right| \\
&= \left| \frac{5(3n^2+4n+4) - 3(5n^2+n+3)}{5(5n^2+n+3)} \right| \\
&= \frac{17n+11}{25n^2+5n+15} \\
&< \frac{17n+11}{25n^2} && \text{since } 25n^2+5n+15 > 25n^2 \\
&\le \frac{28n}{25n^2} && \text{since } 17n+11 \le 17n+11n \\
&= \frac{28}{25n} \\
&< \frac{28}{25N} \\
&= \varepsilon.
\end{aligned}
$$

Theorem If a sequence converges, then its limit is unique.

In other words if (x_n) is a sequence such that

$\lim_{n\to\infty} x_n = L_1$ and $\lim_{n\to\infty} x_n = L_2$, then $L_1 = L_2$.

Lemma If $a, b \in \mathbb{R}$ and $|a-b| < \varepsilon$ for all $\varepsilon > 0$, then $a = b$.

Proof Suppose $a \neq b$. Then $|a-b| > 0$. Let $\varepsilon_0 = \frac{|a-b|}{2}$

Since $|a-b| < \varepsilon$ for all $\varepsilon > 0$, $|a-b| < \varepsilon_0 = \frac{|a-b|}{2}$.

which is a contradiction since it implies $1 < \frac{1}{2}$.

Proof of Theorem To show $L_1 = L_2$, it suffices to show that $|L_1 - L_2| < \varepsilon$ for all $\varepsilon > 0$. Let $\varepsilon > 0$.

Since $x_n \to L_1$, there exists $N_1 \in \mathbb{R}$ such that $|x_n - L_1| < \frac{\varepsilon}{2}$ for all $n > N_1$. Since $x_n \to L_2$, there exists $N_2 \in \mathbb{R}$ such that $|x_n - L_2| < \frac{\varepsilon}{2}$ for all $n > N_2$. Suppose $n > \max\{N_1, N_2\}$. Then

$$\begin{aligned} |L_1 - L_2| &= |L_1 - x_n + x_n - L_2| \\ &\leq |x_n - L_1| + |x_n - L_2| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Theorem If (x_n) converges, then it is bounded.

Proof We must show there exists $M \in \mathbb{R}$ such that $|x_n| \le M$ for all n. Let $L = \lim_{n\to\infty} x_n$. There exists $N \in \mathbb{R}$ such that $|x_n - L| < 1$ for all $n > N$.

Therefore
$$|x_n| = |x_n - L + L|$$
$$\le |x_n - L| + |L|$$
$$< 1 + |L|$$

for all $n > N$. Let $M = \max\{|x_1|, |x_2|, \dots, |x_N|, |L|+1\}$.

Then $|x_n| \le M$ for all n.