Chapter 20 Monotone Convergence Theorem

Def A sequence (x_n) is called monotone if

either (1) it's increasing (i.e. $x_n \le x_{n+1}$ for all n)

or (2) it's decreasing (i.e. $x_n \ge x_{n+1}$ for all n).

Example Let $x_0 = 2$ and define $x_n = \dfrac{x_{n-1}+5}{3}$ for all $n \in \mathbb{Z}^+$

Prove that (x_n) is increasing and bounded above by 3.

Notice $(x_n) = \left(2, \overset{\approx 2.33}{\frac{7}{3}}, \overset{\approx 2.44}{\frac{22}{9}}, \overset{\approx 2.48}{\frac{67}{27}}, \dots\right)$

We'll prove $x_n \le x_{n+1}$ for all $n \in \mathbb{N}$ by induction

The base case, $x_0 \le x_1$, is clear. For the induction step, let $n \in \mathbb{N}$ and suppose $x_n \le x_{n+1}$. We must show $x_{n+1} \le x_{n+2}$. Since $x_n \le x_{n+1}$ adding 5 and dividing by 3 on both sides gives

$$x_{n+1} = \frac{x_n + 5}{3} \le \frac{x_{n+1}+5}{3} = x_{n+2}.$$

Next we'll prove $x_n \le 3$ for all $n \in \mathbb{N}$ by induction.

Observe that the base case, $x_0 \le 3$, is clear.

Let $n \in \mathbb{N}$ and suppose $x_n \le 3$. Then

$$x_{n+1} = \frac{x_n+5}{3} \le \frac{8}{3} < 3$$

Theorem (Monotone Convergence Theorem) Let (x_n) be a monotone sequence. Suppose either

(1) (x_n) is increasing and bounded above

or (2) (x_n) is decreasing and bounded below.

Then (x_n) converges.

Proof under hypothesis (1) Let $L = \sup(x_n)$. We will prove $x_n \to L$. Let $\varepsilon > 0$. Since $L - \varepsilon < L$ we have $L-\varepsilon$ is not an upper bound of (x_n). Therefore there exists $N \in \mathbb{N}$ such that $x_N > L - \varepsilon$. Since (x_n) is increasing, $x_n \geq x_N > L - \varepsilon$ for all $n > N$. Moreover, $x_n \leq L < L + \varepsilon$ for all n. Therefore, for all $n > N$, $|x_n - L| < \varepsilon$.

Example Explain why the sequence in the previous example converges and find its limit.

We've shown the sequence is increasing and bounded above, so it converges by the Monotone Convergence Theorem. Let $L = \lim_{n\to\infty} x_n$. Since $x_{n+1} = \frac{x_n + 5}{3}$ for all $n \in \mathbb{N}$, taking the limit as $n \to \infty$ of both sides gives $L = \frac{L+5}{3}$ which implies $3L = L + 5$ and so $L = 2.5$.