Problem 1. For each sequence (a_n) given below, prove that it converges or diverges.

a. $a_n = 1/\sqrt{n+7}$ for all $n \in \mathbb{N}$

b. $a_n = (n^2+4)/n^2$ for all $n \in \mathbb{Z}^+$

c. $a_n = (3n^2+1)/(4n^2+2)$ for all $n \in \mathbb{N}$

d. $a_n = \cos(n\pi/2)$ for all $n \in \mathbb{N}$

(a) We claim the sequence converges to $L = 0$. Let $\varepsilon > 0$ and define $N = \underline{\varepsilon^{-2}}$. Suppose $n > N$ and observe that

$$\begin{aligned} |a_n - L| &= \left| \frac{1}{\sqrt{n+7}} \right| \\ &< \frac{1}{\sqrt{n}} \\ &< \frac{1}{\sqrt{N}} \\ &= \varepsilon. \end{aligned}$$

(b) We claim the sequence converges to $L = 1$. Let $\varepsilon > 0$ and define $N = \underline{4\varepsilon^{-1/2}}$. Suppose $n > N$ and observe that

$$\begin{aligned} |a_n - L| &= \left| \frac{n^2+4}{n^2} - 1 \right| \\ &= \left| \frac{n^2+4-n^2}{n^2} \right| \\ &= \frac{4}{n^2} \\ &< \frac{4}{N^2} \\ &= \varepsilon. \end{aligned}$$

(c) We claim the sequence converges to $L = \frac{3}{4}$. Let $\varepsilon > 0$ and define $N = \underline{5/8\ \varepsilon^{-1/2}}$. Suppose $n > N$ and observe that

$$
\begin{aligned}
|a_n - L| &= \left| \frac{3n^2+1}{4n^2+2} - \frac{3}{4} \right| \\
&= \left| \frac{(3n^2+4)4 - 3(4n^2+2)}{4(4n^2+2)} \right| \\
&= \frac{10}{4(4n^2+n+2)} \\
&< \frac{5}{8n^2} \\
&< \frac{5}{8N^2} \\
&= \varepsilon.
\end{aligned}
$$

(d) We claim the sequence diverges. Suppose not. Then there exists $L \in \mathbb{R}$ so that there exists $N \in \mathbb{R}$ so that $|a_n - L| < 0.1$ when $n > N$. Note that for any n, $|a_n - a_{n+1}| = 1$. Therefore

$$
\begin{aligned}
1 &= |a_{N+1} - a_{N+2}| \\
&\leq |a_{N+1} - L| + |a_{N+2} - L| \\
&< 0.2,
\end{aligned}
$$

which is a contradiction.

Problem 2. Prove the following by induction:

a. $(3/2)^n > n+1$ for all $n \in \mathbb{N}$ such that $n \geq 5$.

b. 8 divides $5^{2n} - 1$ for all $n \in \mathbb{N}$.

c. $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$ for all $n \in \mathbb{Z}^+$

(a) The base case, when $n=5$, is $(\frac{3}{2})^5 > 6$. Since $(\frac{3}{2})^5 = 7.59375$, this is true. Suppose $n \geq 5$ and suppose $(\frac{3}{2})^n > n+1$. Observe that

$$\left(\frac{3}{2}\right)^{n+1} = \frac{3}{2}\left(\frac{3}{2}\right)^n$$

$$> \frac{3}{2}(n+1)$$

by the induction hypothesis. To complete the proof we must show $\frac{3}{2}(n+1) > n+2$. Notice this is equivalent to $\frac{1}{2}n > \frac{1}{2}$ (by rearranging both sides) which is equivalent to $n > 1$. This is clearly true since $n \geq 5$, so the induction step holds.

(b) The base case, when $n=0$, is $8 \mid 0$. This is true. For the induction step, let $n \in \mathbb{N}$ and suppose $8 \mid (5^{2n} - 1)$. We must show there exists $k \in \mathbb{Z}$ such that $8k = 5^{2(n+1)} - 1$. Note that by the induction hypothesis $5^{2n} = 8j + 1$ for some $j \in \mathbb{Z}$. Let $k = 25j + 3 \in \mathbb{Z}$. Observe that

$$5^{2(n+1)} = 25 \cdot 5^{2n}$$

$$= 25(8j+1)$$

$$= 8(25j+3)+1$$

$$= 8k+1.$$

Thus the induction step holds.

(c) The base case, when $n=1$, is $\sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{2}$.

This is true since the left side contains just one term, namely $\frac{1}{2}$. Next, let $n \in \mathbb{Z}^+$ and suppose $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$ holds. Observe that

$$\sum_{k=1}^{n+1} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \frac{1}{k(k+1)} + \frac{1}{(n+1)(n+2)}$$

$$= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \quad \text{by ind. hypothesis}$$

$$= \frac{n(n+2)+1}{(n+1)(n+2)}$$

$$= \frac{n^2+2n+1}{(n+1)(n+2)}$$

$$= \frac{(n+1)^2}{(n+1)(n+2)}$$

$$= \frac{n+1}{n+2}.$$

Thus the induction step holds.

Problem 3. Compute the following:

a. $\gcd(105, 252)$

b. 2 negative and 2 positive elements of $[27]_5$

c. all solutions $x \in \mathbb{Z}$ of $17 \equiv x \pmod 8$

d. all solutions $x \in \mathbb{Z}^+$, with $x > 1$, of $31 \equiv 15 \pmod x$

(a) $252 = 2 \cdot 105 + 42$

$105 = 2 \cdot 42 + 21$

$42 = 2 \cdot 21 + 0$

So $\gcd(252, 105) = 21$

(b) $[27]_5 = [2]_5 = \{5k + 2 : k \in \mathbb{Z}\}$

So $-8, -3, 2, 7$ are examples

(c) $[17]_8 = [1]_8 = \{8k + 1 : k \in \mathbb{Z}\}$

(d) $31 - 15 = 16$ is divisible by $2, 4, 8, 16$.

Problem 4. True or false:

a. There exists a function $f : \{1, 2, 3\} \to \{1, 3\}$ that is one-to-one.

b. There exists a function $f : \{1, 3\} \to \{1, 2, 3\}$ that is onto.

c. There exists a function $f : \mathbb{R} \to (-\pi/2, \pi/2)$ that is one-to-one.

d. There exists a function $f : (-\pi/2, \pi/2) \to \mathbb{R}$ that is onto.

e. If $f : A \to B$ and $g : B \to C$ are functions such that $g \circ f$ is one-to-one, then g is one-to-one.

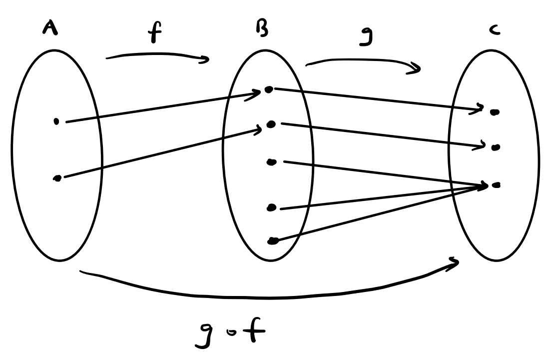
(a) False: for f to be well-defined, every input needs an output but since the domain has 3 elements and the codomain has 2 elements, at least one output value will be repeated.

(b) False: for f to be onto, every codomain value must be mapped to but there are 3 codomai elements and only 2 domain elements

(c) True: $f(x) = \arctan x$

(d) True: $f(x) = \tan x$

(e) False: counterexample:



Problem 5. For each of the following, determine whether $f : A \to B$ is a well-defined function. If so, determine whether it is a bijection. If so, find its inverse. If not, explain why.

a. Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be given by $f(x, y) = (y, x)$.

b. let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be given by $f(x, y) = x^2 + y^2$.

c. Let $y \in \mathbb{R}$ be given and let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = yx$.

d. Let $f : \mathcal{P}(\mathbb{Z}) \to \mathbb{Z}$ be given by $f(S) = \max S$.

(a) well defined, bijection, $f^{-1} = f$

(b) well-defined, not a bijection, not onto :

$\forall\ y \in \mathbb{Z}^-$, $\nexists\ x \in \mathbb{Z} \times \mathbb{Z}$ such that $f(x) = y$

since $f(x) \in \mathbb{N}$

(c) well-defined, bijection if and only if $y \neq 0$

and then $f^{-1}(x) = \frac{1}{y} x$. When $y \neq 0$,

neither one-to-one (since $\forall\ x_1 \neq x_2$, $f(x_1) = f(x_2)$)

nor onto (since $\forall\ y \neq 0$, $\nexists\ x \in \mathbb{R}$ such that

$f(x) = y$ since $f(x) = 0\ \forall\ x \in \mathbb{R}$).

(d) not well-defined : not every element in domain

has an output in $\mathbb{Z}$, eg. $f(\mathbb{N})$ not defined

since $\max \mathbb{N}$ does not exist.