Chapter 6 Sets

Notation	Meaning / Definition
$x \varepsilon A$ or $x \in A$	x is an element of the set A
$x \notin A$ or $x \notin A$	x is <u>not</u> an element of the set A
$A \subseteq B$	A is a <u>subset</u> of B [ie. $\forall x \varepsilon A,\ x \varepsilon B$]
$A \subset B$ or $A \subsetneq B$	A is a <u>proper subset</u> of B [ie. $(\forall x \varepsilon A, x \varepsilon B) \wedge (\exists x \varepsilon B, x \notin A)$]
$A \not\subseteq B$	A is <u>not</u> a subset of B [ie. $\exists x \varepsilon A,\ x \notin B$]
$A = B$	A <u>equals</u> B [ie. $(\forall x \varepsilon A, x \varepsilon B) \wedge (\forall x \varepsilon B, x \varepsilon A)$]
ϕ	<u>empty set</u> (ie. the set with no elements)

Example Let $S = \{2n+1 : n \in \mathbb{Z}\}$ and $T = \{s^2 : s \in S\}$.

Prove that $T \subsetneq S$.

Proof We begin by showing the inclusion $T \subset S$.

Let $x \in T$. Then there exists $s \in S$ such that $x = s^2$.

We must now show $x \in S$. Note that since $s \in S$,

there exists $n \in \mathbb{Z}$ so that $s = 2n+1$. Therefore

$$
\begin{aligned}
x &= s^2 \\
&= (2n+1)^2 \\
&= 4n^2 + 4n + 1 \\
&= 2(2n^2 + 2n) + 1.
\end{aligned}
$$

Thus, since $2n^2 + 2n \in \mathbb{Z}$, $x \in S$.

Now, to show $T \subsetneq S$, we must show there exists

$x \in S$ such that $x \notin T$. Indeed, notice $x = 5$

is in S since $5 = 2n+1$ when $n = 2$. Also, $5 \notin T$

since its square root is not an odd integer.

<u>Example</u> Let $A = \{(x,y) \in \mathbb{R}^2 : x^2 - x = y^2 = 0\}$ and $B = \{(0,0), (1,0)\}$. Prove $A = B$.

<u>Proof</u> Let $(u,v) \in A$. Then $u^2 - u = 0$ and $v^2 = 0$. Therefore $u(u-1) = 0$ and $v = 0$. Since $u(u-1) = 0$, $u = 0$ or $u = 1$, and in either case $v = 0$. Thus $(u,v) = (0,0)$ or $(u,v) = (1,0)$, so $(u,v) \in B$. Therefore $A \subseteq B$.

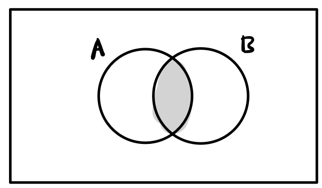
Now, we'll show $B \subseteq A$ by showing for every $(u,v) \in B$, we have that $(u,v) \in A$. If $(u,v) = (0,0)$, then $u^2 - u = 0^2 - 0 = 0$ and $v^2 = 0^2 = 0$. Therefore $(u,v) \in A$ in this case. If $(u,v) = (1,0)$, then $u^2 - u = 1^2 - 1 = 0$ and $v^2 = 0^2 = 0$. Thus $(u,v) \in A$ in this case.

We conclude that, since $A \subseteq B$ and $B \subseteq A$, we have $A = B$.

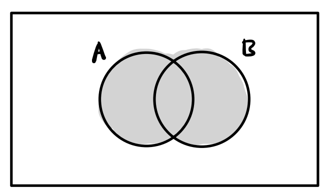
$A \cap B$ the set of elements common to A <u>and</u> B

[ie. $x \in A \cap B$ if $(x \in A) \wedge (x \in B)$]

(<u>intersection</u> of A and B)



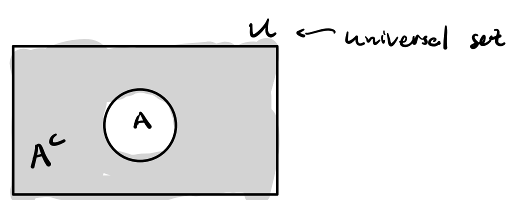
$A \cup B$ the set of elements in A <u>or</u> B (<u>union</u>)



(ie. $x \in A \cup B$ if $(x \in A) \vee (x \in B)$)

A^c the set of elements <u>not</u> in A (<u>complement</u>)

(ie. $x \in A^c$ if $\neg(x \in A)$.)



$A \setminus B$ the set of elements in A but not in B

(ie. $x \in A \setminus B$ if $(x \in A) \wedge (x \notin B)$)

(<u>set difference</u>) note this is $A \cap B^c$

Let $A = \{1,2,3,4\}, B = \{1,2,3,4,5,6\}, S = \{2n : n \in \mathbb{Z}\}$.

Problem 1. Which of the following are true mathematical statements?

a. $3 \in A$

b. $\{3\} \in A$

c. $5 \in A$

d. $2 \in a$

e. $2 \notin A$

f. $\emptyset \in A$

a is the only true statement

Problem 2. Which of the following are true?

a. $A \subseteq B$

b. $B \subseteq A$

c. $A \in B$

d. $\{3\} \subseteq A$

e. $S \not\subseteq \mathbb{Z}$

f. $S \subsetneq \mathbb{Z}$

g. $\emptyset \subseteq A$

a, d, f, g are true

Problem 3. Let $C = \{x \in \mathbb{N} : x = y^2 \text{ for some } y \in \mathbb{R}\}$. Prove that $C = \mathbb{N}$.

We will show $C \subseteq \mathbb{N}$ and $\mathbb{N} \subseteq C$. In fact $C \subseteq \mathbb{N}$ is clear since the definition of C requires that every element of C be a natural number. To show $\mathbb{N} \subseteq C$, suppose $x \in \mathbb{N}$. We must show there exists $y \in \mathbb{R}$ such that $x = y^2$. Let $y = \sqrt{x}$. Then $y \in \mathbb{R}$ and $y^2 = x$. Therefore, $x \in C$, and so $\mathbb{N} \subseteq C$.

Problem 4. Let $D = \{(2n+1)^3 : n \in \mathbb{Z}\}, E = \{2n+1 : n \in \mathbb{Z}\}$. Prove that $D \subsetneq E$.

We begin by showing $D \subset E$. We must show that for all $x \in D$, we have $x \in E$. Let $x \in D$. Then

$x = (2n+1)^3$ for some $n \in \mathbb{Z}$. Observe that

$$\begin{aligned} x &= (2n+1)(4n^2+4n+1) \\ &= 8n^3+12n^2+6n+1 \\ &= 2(4n^3+6n^2+3n)+1. \end{aligned}$$

Therefore, since $m = 4n^3+6n^2+3n \in \mathbb{Z}$, we have $x \in E$.

Now, to show $D \subsetneq E$, we must show there exists $x \in E$ such that $x \notin D$. In words, we must show there is an odd integer which is not the cube of an odd integer. We claim $9 \in E$ but $9 \notin D$. It's indeed the case that $9 \in E$ since $9 = 2(4)+1$ and $4 \in \mathbb{Z}$. To show $9 \notin D$, suppose by way of contradiction that $9 \in D$. Then $9 = (2n+1)^3$ for some $n \in \mathbb{Z}$. Then $n = \frac{1}{2}(9^{1/3}-1)$ but this is a contradiction since it is not an integer.