Chapter 7 Operations on Sets

Today we'll talk about formulas involving set operations and the general strategy for proofs of such formulas. The worksheet (and textbook Theorem 7.4) has a big list of formulas. We'll warm up with a proof of one of them.

Theorem (distributive property) Let A, B, and C be sets. Then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof. We'll prove this by showing two set containments: $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ and $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. For the first, let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. In the first case, $x \in A$ implies $x \in A \cup B$ and $x \in A \cup C$.

Therefore $x \in (A \cup B) \cap (A \cup C)$. In the second case, $x \in B \cap C$ implies $x \in B$ and $x \in C$. Therefore $x \in A \cup B$ and $x \in A \cup C$, so $x \in (A \cup B) \cap (A \cup C)$. Together these cases prove that

$$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C).$$

Now suppose $x \in (A \cup B) \cap (A \cup C)$. Therefore $x \in A \cup B$ and $x \in A \cup C$. [Key step:] We now break our proof into two cases: $x \in A$ or $x \notin A$. If $x \in A$, then clearly $x \in A \cup (B \cap C)$. Suppose $x \notin A$. Then $x \in A \cup B$ and $x \notin A$ implies $x \in B$. Moreover, $x \in A \cup C$ and $x \notin A$ implies $x \in C$. Thus in this case we have $x \in B \cap C$, which implies $x \in A \cup (B \cap C)$. Togthather these cases prove that

$$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C).$$