Chapter 8 More operations on sets.

Def Let A, B be sets. Then their union $A \cup B$ is a set such that $x \in A \cup B$ if $x \in A$ or $x \in B$.

Their intersection $A \cap B$ is the set such that $x \in A \cap B$ if $x \in A$ and $x \in B$.

More generally, we can consider a collection (ie set) $\mathcal{A}$ of subsets of our universal set and the union or intersection of all elements of this collection:

$$\bigcup_{A \in \mathcal{A}} A = \{ x : x \in A \text{ for some } A \in \mathcal{A} \}$$

$$\bigcap_{A \in \mathcal{A}} A = \{ x : x \in A \text{ for all } A \in \mathcal{A} \}.$$

We often express our collections by labeling each element. Suppose there is a set I, called an index set, such that for each $\alpha \in I$, there corresponds a unique $A_\alpha \in \mathcal{A}$. Then we write $\mathcal{A} = \{ A_\alpha : \alpha \in I \}$

and $\bigcup_{\alpha \in I} A_\alpha$ or $\bigcap_{\alpha \in I} A_\alpha$ for the union

or intersection over the collection.

<u>Examples</u> Simplify the following

① $\bigcup_{x \in \mathbb{R}^+} (0, x) = (0, \infty)$

← note $\mathbb{R}^+ = (0, \infty)$

② $\bigcup_{n \in \mathbb{N}} [0, n] = \bigcup_{n=0}^{\infty} [0, n] = \{0\} \cup [0,1] \cup [0,2] \cup \cdots$

$= [0, \infty)$

③ $\bigcap_{n \in \mathbb{N}} [0, n] = \bigcap_{n=0}^{\infty} [0, n] = \{0\} \cap [0,1] \cap [0,2] \cap \cdots$

$= \{0\}$

Theorem (DeMorgan's Laws) Let X be a universal set, I an index set and $\mathcal{A} = \{A_\alpha \subseteq X : \alpha \in I\}$ a collection.

Then

① $X \setminus \bigcup_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I} (X \setminus A_\alpha)$

② $X \setminus \bigcap_{\alpha \in I} A_\alpha = \bigcup_{\alpha \in I} (X \setminus A_\alpha)$

Proof of ① We prove a double inclusion.

Let $x \in X \setminus \bigcup_{\alpha \in I} A_\alpha$. Then $x \in X$ and $x \notin \bigcup_{\alpha \in I} A_\alpha$.

Since $x \notin \bigcup_{\alpha \in I} A_\alpha$, for all $\alpha \in I$, $x \notin A_\alpha$. Therefore, for all $\alpha \in I$, $x \in X \setminus A_\alpha$, which implies $x \in \bigcap_{\alpha \in I} (X \setminus A_\alpha)$.

Thus $X \setminus \bigcup_{\alpha \in I} A_\alpha \subseteq \bigcap_{\alpha \in I} (X \setminus A_\alpha)$.

Let $x \in \bigcap_{\alpha \in I} (X \setminus A_\alpha)$. Then for all $\alpha \in I$, $x \in X \setminus A_\alpha$, which means $x \in X$ and for all $\alpha \in I$, $x \notin A_\alpha$.

Thus $x \in X$ and $x \notin \bigcup_{\alpha \in I} A_\alpha$, and so

$x \in X \setminus \bigcup_{\alpha \in I} A_\alpha$. Therefore

$$\bigcap_{\alpha \in I} (X \setminus A_\alpha) \subseteq X \setminus \bigcup_{\alpha \in I} A_\alpha$$

Problem 1. For each positive integer n, define

$$A_n = [0, 1/n), \quad B_n = [0, 1/n], \quad C_n = (0, 1/n).$$

Find the following:

a. $\bigcup_{n=1}^{\infty} A_n, \bigcup_{n=1}^{\infty} B_n, \bigcup_{n=1}^{\infty} C_n$

b. $\bigcap_{n=1}^{\infty} A_n, \bigcap_{n=1}^{\infty} B_n, \bigcap_{n=1}^{\infty} C_n$

(a) $$\bigcup_{n=1}^{\infty} A_n = [0,1) \cup [0, \tfrac{1}{2}) \cup [0, \tfrac{1}{3}) \cup \cdots = [0,1)$$

$$\bigcup_{n=1}^{\infty} B_n = [0,1] \cup [0, \tfrac{1}{2}] \cup [0, \tfrac{1}{3}] \cup \cdots = [0,1]$$

$$\bigcup_{n=1}^{\infty} C_n = (0,1) \cup (0, \tfrac{1}{2}) \cup (0, \tfrac{1}{3}) \cup \cdots = (0,1)$$

(b) $$\bigcap_{n=1}^{\infty} A_n = [0,1) \cap [0, \tfrac{1}{2}) \cap [0, \tfrac{1}{3}) \cap \cdots = \{0\}$$

$$\bigcap_{n=1}^{\infty} B_n = [0,1] \cap [0, \tfrac{1}{2}] \cap [0, \tfrac{1}{3}] \cap \cdots = \{0\}$$

$$\bigcap_{n=1}^{\infty} C_n = (0,1) \cap (0, \tfrac{1}{2}) \cap (0, \tfrac{1}{3}) \cap \cdots = \phi$$

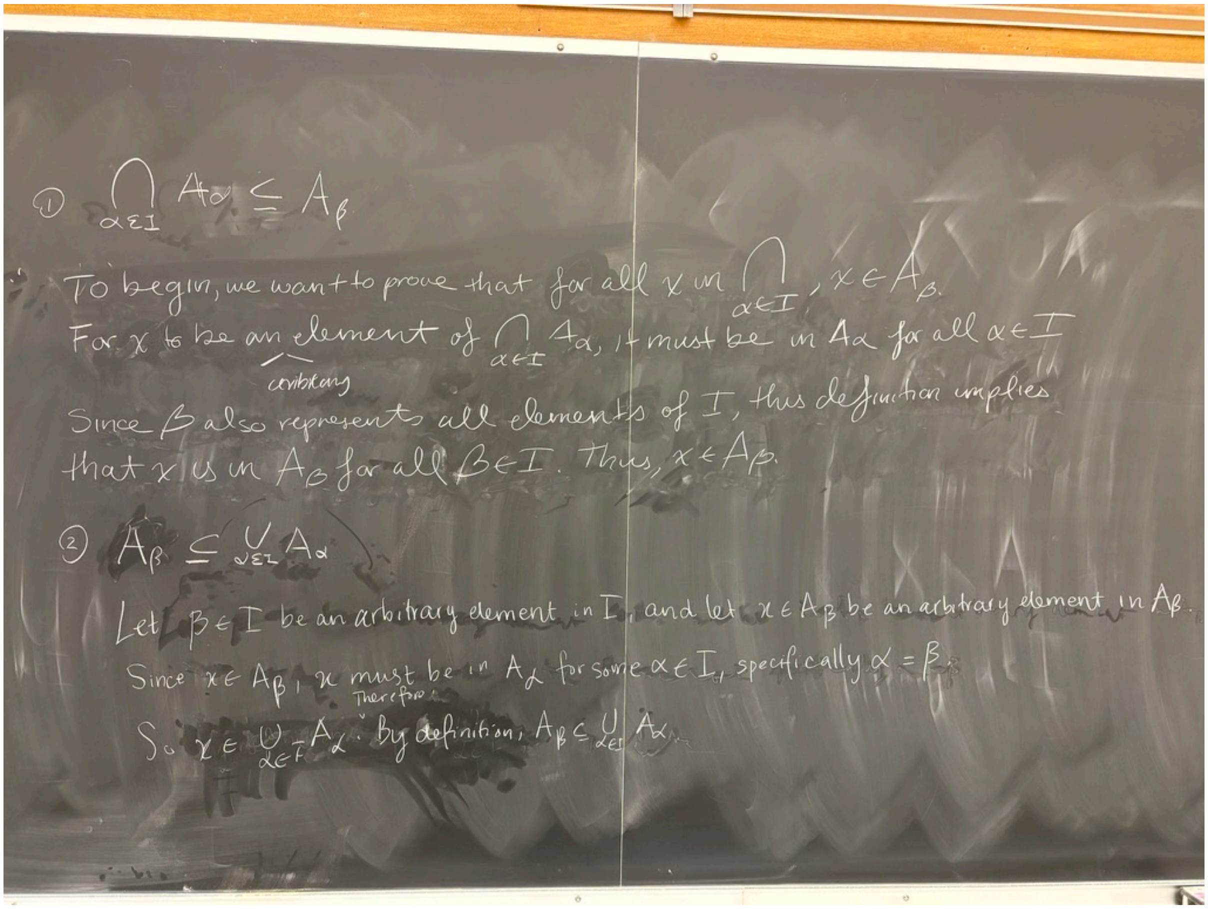
Problem 2. Simplify the following set:

$$\bigcap_{j \in \mathbb{Z}} (\mathbb{R} \setminus (j, j+1)).$$

$$= \mathbb{R} \setminus \bigcup_{j \in \mathbb{Z}} (j, j+1)$$

$$= \mathbb{R} \setminus [\cdots \cup (-2,-1) \cup (-1,0) \cup (0,1) \cup (1,2) \cdots]$$

$$= \mathbb{Z}$$



Let $\beta \in I$. We first show $\bigcap_{\alpha\in I} A_\alpha \subseteq A_\beta$. Let $x \in \bigcap_{\alpha\in I} A_\alpha$. Then $x \in A_\alpha$ for all $\alpha \in I$. In particular, we have $x \in A_\beta$ since $\beta \in I$.

Next we show $A_\beta \subseteq \bigcup_{\alpha\in I} A_\alpha$. Let $x \in A_\beta$. To show $x \in \bigcup_{\alpha\in I} A_\alpha$ we must show $x \in A_\alpha$ for some $\alpha \in I$. This is indeed the case for $\alpha = \beta$.