

Chapter 8 More operations on sets.

Def Let A, B be sets. Then their union $A \cup B$ is a set such that $x \in A \cup B$ if $x \in A$ or $x \in B$.

Their intersection $A \cap B$ is the set such that $x \in A \cap B$ if $x \in A$ and $x \in B$.

More generally, we can consider a collection (ie set) \mathcal{A} of subsets of our universal set and the union or intersection of all elements of this collection:

$$\bigcup_{A \in \mathcal{A}} A = \{x : x \in A \text{ for some } A \in \mathcal{A}\}$$

$$\bigcap_{A \in \mathcal{A}} A = \{x : x \in A \text{ for all } A \in \mathcal{A}\}.$$

We often express our collection by labeling each element. Suppose there is a set I , called an index set, such that for each $\alpha \in I$, there corresponds a unique $A_\alpha \in \mathcal{A}$. Then we write $\mathcal{A} = \{A_\alpha : \alpha \in I\}$

and $\bigcup_{\alpha \in I} A_\alpha$ or $\bigcap_{\alpha \in I} A_\alpha$ for the union

or intersection over the collection.

Examples Simplify the following

$$\textcircled{1} \quad \bigcup_{x \in \mathbb{R}^+} (0, x) = (0, \infty)$$

← note $\mathbb{R}^+ = (0, \infty)$

$$\textcircled{2} \quad \bigcup_{n \in \mathbb{N}} [0, n] = \bigcup_{n=0}^{\infty} [0, n] = \{0\} \cup [0, 1] \cup [0, 2] \cup \dots = [0, \infty)$$

$$\textcircled{3} \quad \bigcap_{n \in \mathbb{N}} [0, n] = \bigcap_{n=0}^{\infty} [0, n] = \{0\} \cap [0, 1] \cap [0, 2] \cap \dots = \{0\}$$

Theorem (DeMorgan's Law.) Let X be a universal set,

I an index set and $A = \{A_\alpha \subseteq X : \alpha \in I\}$ a collection.

Then $\textcircled{1} \quad X \setminus \bigcup_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I} (X \setminus A_\alpha)$

$$\textcircled{2} \quad X \setminus \bigcap_{\alpha \in I} A_\alpha = \bigcup_{\alpha \in I} (X \setminus A_\alpha)$$

Proof of $\textcircled{1}$ We prove a double inclusion.

Let $x \in X \setminus \bigcup_{\alpha \in I} A_\alpha$. Then $x \in X$ and $x \notin \bigcup_{\alpha \in I} A_\alpha$.

Since $x \notin \bigcup_{\alpha \in I} A_\alpha$, for all $\alpha \in I$, $x \notin A_\alpha$. Therefore,

for all $\alpha \in I$, $x \in X \setminus A_\alpha$, which implies $x \in \bigcap_{\alpha \in I} (X \setminus A_\alpha)$.

Thus $X \setminus \bigcup_{\alpha \in I} A_\alpha \subseteq \bigcap_{\alpha \in I} (X \setminus A_\alpha)$.

Let $x \in \bigcap_{\alpha \in I} (X \setminus A_\alpha)$. Then for all $\alpha \in I$, $x \in X \setminus A_\alpha$,

which means $x \in X$ and for all $\alpha \in I$, $x \notin A_\alpha$.

Thus $x \in X$ and $x \notin \bigcup_{\alpha \in I} A_\alpha$, and so

$x \in X \setminus \bigcup_{\alpha \in I} A_\alpha$. Therefore

$$\bigcap_{\alpha \in I} (X \setminus A_\alpha) \subseteq X \setminus \bigcup_{\alpha \in I} A_\alpha$$

Problem 1. For each positive integer n , define

$$A_n = [0, 1/n], \quad B_n = [0, 1/n], \quad C_n = (0, 1/n).$$

Find the following:

- a. $\bigcup_{n=1}^{\infty} A_n, \bigcup_{n=1}^{\infty} B_n, \bigcup_{n=1}^{\infty} C_n$
- b. $\bigcap_{n=1}^{\infty} A_n, \bigcap_{n=1}^{\infty} B_n, \bigcap_{n=1}^{\infty} C_n$

$$\textcircled{a} \quad \bigcup_{n=1}^{\infty} A_n = [0, 1) \cup [0, \frac{1}{2}) \cup [0, \frac{1}{3}) \cup \dots = [0, 1)$$

$$\bigcup_{n=1}^{\infty} B_n = [0, 1] \cup [0, \frac{1}{2}] \cup [0, \frac{1}{3}] \cup \dots = [0, 1]$$

$$\bigcup_{n=1}^{\infty} C_n = (0, 1) \cup (0, \frac{1}{2}) \cup (0, \frac{1}{3}) \cup \dots = (0, 1)$$

$$\textcircled{b} \quad \bigcap_{n=1}^{\infty} A_n = [0, 1] \cap [0, \frac{1}{2}] \cap [0, \frac{1}{3}] \cap \dots = \{0\}$$

$$\bigcap_{n=1}^{\infty} B_n = [0, 1] \cap [0, \frac{1}{2}] \cap [0, \frac{1}{3}] \cap \dots = \{0\}$$

$$\bigcap_{n=1}^{\infty} C_n = (0, 1) \cap (0, \frac{1}{2}) \cap (0, \frac{1}{3}) \cap \dots = \emptyset$$

Problem 2. Simplify the following set:

$$\bigcap_{j \in \mathbb{Z}} (\mathbb{R} \setminus (j, j+1)).$$

$$= \mathbb{R} \setminus \bigcup_{j \in \mathbb{Z}} (j, j+1)$$

$$= \mathbb{R} \setminus [\dots \cup (-2, -1) \cup (-1, 0) \cup (0, 1) \cup (1, 2) \dots]$$

$$= \mathbb{Z}$$

$$\textcircled{1} \quad \bigcap_{\alpha \in I} A_\alpha \subseteq A_\beta$$

To begin, we want to prove that for all x in $\bigcap_{\alpha \in I} A_\alpha$, $x \in A_\beta$.

For x to be an element of $\bigcap_{\alpha \in I} A_\alpha$, it must be in A_α for all $\alpha \in I$ (arbitrary).

Since β also represents all elements of I , this definition implies that x is in A_β for all $\beta \in I$. Thus, $x \in A_\beta$.

$$\textcircled{2} \quad A_\beta \subseteq \bigcup_{\alpha \in I} A_\alpha$$

Let $\beta \in I$ be an arbitrary element in I , and let $x \in A_\beta$ be an arbitrary element in A_β .

Since $x \in A_\beta$, x must be in A_α for some $\alpha \in I$, specifically $\alpha = \beta$ (Therefore).

So, $x \in \bigcup_{\alpha \in I} A_\alpha$. By definition, $A_\beta \subseteq \bigcup_{\alpha \in I} A_\alpha$.

Let $\beta \in I$. We first show $\bigcap_{\alpha \in I} A_\alpha \subseteq A_\beta$. Let

$x \in \bigcap_{\alpha \in I} A_\alpha$. Then $x \in A_\alpha$ for all $\alpha \in I$. In

particular, we have $x \in A_\beta$ since $\beta \in I$.

Next we show $A_\beta \subseteq \bigcup_{\alpha \in I} A_\alpha$. Let $x \in A_\beta$

To show $x \in \bigcup_{\alpha \in I} A_\alpha$ we must show $x \in A_\alpha$ for

some $\alpha \in I$. This is indeed the case for $\alpha = \beta$.