Math 206 — Exam 2 review guide

Your exam in class on November 8 will contain about 5 problems, some with multiple parts. You should expect to see a question asking you to state some definitions or theorems, a question asking you to prove a statement that was proved in class, and a few questions in the style similar to worksheets, homework, and quizzes. You should expect to write 2-3 proofs. It will cover material from Homework 5 to Homework 7. In the textbook, this is material spanning Chapters 9-14, along with some material on open and closed sets. I have outlined some important definitions, theorems, and genearl topics below. Also, the problems below give you a sampling of some problems like those that will appear on the exam, but it's not necessarily comprehensive, so make sure to review old homework, quizzes, worksheets, and lecture notes.

Definitions and theorems

While not necessarily comprehensive, here is an absolutely-must-know list of definitions.

- open set in $\mathbb{R}$, closed set in $\mathbb{R}$
- power set of a set; Cartesian product of two sets
- relation, equivalence relation, equivalence class; reflexive, symmetric, transitive properties of relations
- partition of a set
- upper bound, lower bound of a set; bounded above, bounded below, bounded set; supremum, infimum, maximum, minimum of a set
- complete set
- function, domain, codomain, range

Similarly, here is an absolutely-must-know list of statements you should know.

- Arbitrary unions and finite intersections of open sets are open; arbitrary intersections and finite unions of closed sets are closed.
- Completeness Axiom of $\mathbb{R}$; Archimedean property of $\mathbb{R}$; Well-ordering principle of $\mathbb{N}$; Between any two real numbers there is a rational number (aka $\mathbb{Q}$ is dense in $\mathbb{R}$).
- The equivalence classes of an equivalence relation form a partition; a partition gives rise to an equivalence relation.

Proofs and logic

You should know the following about proofs and logic. You should know how to:

- Prove a subset of $\mathbb{R}$ is closed using theorems about unions and intersections of open sets and DeMorgan's laws.

- Prove a given relation is or is not an equivalence relation.
- Find the equivalence class of an element for a given equivalence relation.
- Prove a given collection of sets is a partition.
- Find the supremum of a set that is bounded above and prove it; likewise for infimum.
- Find the range of a function and prove it.

Sample problems

These problems are not comprehensive and there are more than you will see on the exam itself, but will give you an idea of the kinds of proof questions to expect.

Problem 1. Give a short proof that the set $\mathbb{Z}$ is a closed set in $\mathbb{R}$ using known results about open sets. State clearly any known results used.

Problem 2. Let $\{B_n : n \in \mathbb{Z}^+\}$ be a given collection of sets. Prove that $\mathcal{P}\left(\bigcap_{n=1}^{\infty} B_n\right) = \bigcap_{n=1}^{\infty} \mathcal{P}(B_n)$.

Problem 3. If $a, b \in \mathbb{R}$, say that $a \sim b$ if and only if $a^k = b^k$ for some positive integer k. Prove that this is an equivalence relation on $\mathbb{R}$ or explain which properties hold and which fail. Repeat the question if the relation is changed to $a \sim b$ if and only if $a = b^k$ for some positive integer k.

Problem 4. Let $A_r = \{x \in \mathbb{R} : |x| = r\}$ for each $r \in \mathbb{R}$. Consider the collection $\mathcal{A} = \{A_r : r \in \mathbb{R}\}$. Is $\mathcal{A}$ a partition of $\mathbb{R}$? Explain which properties of a partition hold, if any, and which fail, if any.

Problem 5. Find the supremum of $A = \{2 - 3/n : n \in \mathbb{Z}^+\}$ and prove that your value is correct. Does this set have a maximum or minimum? Find them if so.

Problem 6. Let $f : A \to \mathbb{R}$ be given by $f(x) = \sqrt{x^2 - 4}$ where $A \subseteq \mathbb{R}$ is the largest set which is a valid domain of f.

a. Express A as an interval or union of intervals.

b. Prove that the range of f is $[0, \infty)$.