Math 206 — Exam 3 review guide

Your exam during the self-scheduled exam period, December 13-17, will contain about 5 problems, some with multiple parts. You should expect to see a question asking you to state some definitions or theorems, a question asking you to prove a statement that was proved in class, and a few questions in the style similar to worksheets, homework, and quizzes. You should expect to write 2-3 proofs. It will cover material from Homework 8 to Homework 10 and modular arithmetic, which was not part of homework. In the textbook, this is material spanning Chapters 15, 16, 18, 19, 20, 27. I have outlined some important definitions, theorems, and genearl topics below. Also, the problems below give you a sampling of some problems like those that will appear on the exam, but it's not necessarily comprehensive, so make sure to review old homework, quizzes, worksheets, and lecture notes.

Definitions and theorems

While not necessarily comprehensive, here is an absolutely-must-know list of definitions.

- one-to-one, onto, and bijective functions; restriction of a function; composition of functions; inverse of a bijective function; identity function
- sequence; increasing/decreasing/strictly increasing/strictly decreasing sequence; a sequence that is bounded above/bounded below/bounded; an upper bound/lower bound/supremum/infimum of a sequence
- convergent/divergent sequence, monotone sequence
- congruence modulo n; integers modulo n; greatest common divisor

Similarly, here is an absolutely-must-know list of statements you should know.

- Principle of Mathematical Induction; Second Principle of Mathematical Induction (Strong Induction)
- Algebraic Limit Theorems for sequences
- Monotone Convergence Theorem
- Convergent sequences are bounded
- Division Algorithm

Proofs and logic

You should know the following about proofs and logic. You should know how to:

- Prove a function is or is not one-to-one, onto, or bijective, and how to find its inverse if it is bijective.
- Use mathematical induction to prove things like formulas or inequalities.

- Prove sequences are bounded or find their infima or suprema.
- Prove sequences converge or diverge.
- Use the Division Algorithm to understand what elements make up $\mathbb{Z}_n$.
- Use the Euclidean Algorithm to compute greatest common divisors.

Sample problems

These problems are not comprehensive and there are more than you will see on the exam itself, but will give you an idea of the kinds of proof questions to expect.

Problem 1. For each sequence (a_n) given below, prove that it converges or diverges.

a. $a_n = 1/\sqrt{n+7}$ for all $n \in \mathbb{N}$

b. $a_n = (n^2+4)/n^2$ for all $n \in \mathbb{Z}^+$

c. $a_n = (3n^2+1)/(4n^2+2)$ for all $n \in \mathbb{N}$

d. $a_n = \cos(n\pi/2)$ for all $n \in \mathbb{N}$

Problem 2. Prove the following by induction:

a. $(3/2)^n > n+1$ for all $n \in \mathbb{N}$ such that $n \geq 5$.

b. 8 divides $5^{2n} - 1$ for all $n \in \mathbb{N}$.

c. $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$ for all $n \in \mathbb{Z}^+$

Problem 3. Compute the following:

a. $\gcd(105, 252)$

b. 2 negative and 2 positive elements of $[27]_5$

c. all solutions $x \in \mathbb{Z}$ of $17 \equiv x \pmod 8$

d. all solutions $x \in \mathbb{Z}^+$, with $x > 1$, of $31 \equiv 15 \pmod x$

Problem 4. True or false:

a. There exists a function $f : \{1,2,3\} \to \{1,3\}$ that is one-to-one.

b. There exists a function $f : \{1,3\} \to \{1,2,3\}$ that is onto.

c. There exists a function $f : \mathbb{R} \to (-\pi/2, \pi/2)$ that is one-to-one.

d. There exists a function $f : (-\pi/2, \pi/2) \to \mathbb{R}$ that is onto.

e. If $f : A \to B$ and $g : B \to C$ are functions such that $g \circ f$ is one-to-one, then g is one-to-one.

Problem 5. For each of the following, determine whether $f : A \to B$ is a well-defined function. If so, determine whether it is a bijection. If so, find its inverse. If not, explain why.

a. Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be given by $f(x,y) = (y,x)$.

b. let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be given by $f(x,y) = x^2 + y^2$.

c. Let $y \in \mathbb{R}$ be given and let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = yx$.

d. Let $f : \mathcal{P}(\mathbb{Z}) \to \mathbb{Z}$ be given by $f(S) = \max S$.