Math 241, Spring 2022 — Homework 5

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Due March 2 at 5:00 pm

Instructions. This problem set covers material from Week 5 of class.

Problem 1. Watch the following Numberphile video that discusses the logistic map, bifurcations, and something called Figenbaum's constant.

1. What ideas were you already comfortable with and what felt new?
2. Explain the ratio that is described by Feigenbaum's constant.
3. Have a friend or family member watch the video too. What came up as you talked about the ideas in the video? What were the two of you left wondering about?

Problem 2. Consider the family of functions $F_\lambda(x) = x + x^2 + \lambda$.

1. Use Desmos to make a conjecture for a value of λ_0 that yields a bifurcation where the number of fixed points in the system changes. State your conjectured value of λ_0 and explain how many fixed points the system has when (1) $\lambda < \lambda_0$, (2) $\lambda = \lambda_0$, and (3) $\lambda > \lambda_0$.
2. Use algebra to find formulas for the fixed points in each range described in part 1.
3. For each fixed point, give the range of values of λ when it is attracting.

Problem 3. Consider the same family of functions as in Problem 2.

1. Use Desmos to make three plots of $F_\lambda^2(x)$ along with $y = x$ for three different values of λ: $\lambda = -0.8, \lambda = -1, \lambda = -1.3$. For each plot, state how many intersection points there are between $F_\lambda^2(x)$ and the line $y = x$, and explain whether they are period-2 points or fixed points.
2. Summarize in a sentence or two what is going on around the value $\lambda = -1$. What kind of bifurcation is this called?
3. What role did $\lambda = -1$ play in relation to your work in Problem 2 with fixed points?

Problem 4. Consider the family of functions $F_\lambda(x) = \lambda x + x^3$.

1. Use Desmos to make a conjecture for a value of λ_0 that yields a bifurcation where the number of fixed points in the system changes. State your conjectured value of λ_0 and explain how many fixed points the system has when (1) $\lambda < \lambda_0$, (2) $\lambda = \lambda_0$, and (3) $\lambda > \lambda_0$.
2. Use algebra to find formulas for the fixed points in each range described in part 1.
3. For each fixed point, give the range of values when it is attracting.

Problem 5. Consider the same family of functions as in Problem 4.

1. Use Desmos to make three plots of $F_\lambda^2(x)$ along with $y = x$ for three different values of λ: $\lambda = -0.8, \lambda = -1, \lambda = -1.3$. For each plot, state how many intersection points there are between $F_\lambda^2(x)$ and the line $y = x$, and explain whether they are period-2 points or fixed points.
2. Summarize in a sentence or two what is going on around the value $\lambda = -1$. What kind of bifurcation is this called?
3. What role did $\lambda = -1$ play in relation to your work in Problem 4 with fixed points?
4. Use Desmos to describe in a few sentences the bifurcation that occurs at $\lambda = -2$.