Math 241, Spring 2022 — Homework 6

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Due March 30 at 5:00 pm

Instructions. This problem set covers material from Week 9 of class.

Problem 1. The following questions are about the logistic map $F_\lambda(x) = \lambda x(1-x)$ and mimic the ideas we discussed about the quadratic map $Q_c(x) = x^2 + c$ when $c < -2$.

1. Make a plot with a slider of $F_\lambda(x)$ in Desmos along with the the lines $y = 0, y = 1, x = 0, x = 1$. What do you notice about the graph of $F_\lambda(x)$ in relation to the square bounded by the 4 given lines, particularly when $\lambda > 4$.
2. Suppose $\lambda > 4$ and let $I = [0, 1]$.
   (a) If $x_0 \notin I$, what can you say about the orbit of x_0? Does any iterate of x_0 enter I? Justify your claim graphically and describe the behavior of iterates of x_0 in words.
   (b) Let $A_1 \subseteq I$ be the set of initial seeds whose orbit leaves I after 1 iteration of F_λ. That is, A_1 consists of $x_0 \in I$ such that $F_\lambda(x_0) \notin I$. Make a sketch in the xy-plane of $F_\lambda(x)$, label the set A_1 in your graph, and give a brief justification in words.
   (c) Repeat the previous part with the set $A_2 \subseteq I$ of initial seeds whose orbit leaves I after 2 iterations of F_λ.
   (d) One more: repeat the previous part with the set $A_3 \subseteq I$ of initial seeds whose orbit leaves I after 3 iterations of F_λ.
3. Like in the previous part, suppose $\lambda > 4$ and continue with established notation. Let $\Lambda = I - \bigcup_{n=1}^{\infty} A_n$ be the set of initial seeds x_0 with the property $F_\lambda^n(x_0) \in I$ for all n. Your previous work might make it feel like Λ is actually an empty set, but explain why Λ is non-empty. More precisely, explain why it actually contains infinitely many points.
4. *Challenge problem (not for credit)*: Find a value $\lambda_0 > 4$ so that whenever $\lambda > \lambda_0$ we have $F_\lambda'(x) > 1$ for all $x \in I - A_1$. Use this to explain, at least in the case where $\lambda > \lambda_0$, why the set Λ contains no intervals, mimicking the discussion in Section 7.2.

Problem 2. Do the following exercises from Chapter 7, page 89.

1. Exercise 9. Note that you can use the following syntax in Desmos to make the piecewise graph: $T(x) = \left\{x \leq \frac{1}{2} : 3x, x > \frac{1}{2} : 3 - 3x\right\}$.
2. Exercise 12. You may give a graphical analysis argument to do this problem. It's helpful to plot the box bounded by the lines $y = 0, y = 1, x = 0, x = 1$ too.
3. Exercise 13. Again, you may give a graphical analysis argument. It might be helpful to plot $T^2(x)$ in Desmos to see the significance of the given intervals.
4. What do you think can be said about the orbits of initial seeds x_0 in the Cantor middle thirds set under iteration of T?

Problem 3. Do the following exercises from Chapter 7, page 89.

1. Exercise 3
2. Exercise 5
3. Exercise 6