Math 241, Spring 2022 — Homework 8

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Due April 13 at 5:00 pm

Instructions. This problem set covers material from Week 11 of class.

Problem 1. Let K be the Cantor middle-thirds set. In class we learned that when $x \in K$ the ternary expansion of

$$T(x) = \begin{cases} 3x & x \leq 1/2 \\ 3 - 3x & x > 1/2 \end{cases}$$

is closely related to the ternary expansion of x. In particular, we learned that if $x = 0.0s_2s_3s_4\ldots$ then $T(x) = 0.s_2s_3s_4\ldots$ and if $x = 0.2s_2s_3s_4\ldots$ then $T(x) = 0.(2-s_2)(2-s_3)(2-s_4)\ldots$.

1. Let $x \in K$ be the value with ternary expansion $0.\overline{020}$. Give the ternary expansions of $T(x), T^2(x)$, and $T^3(x)$ and describe what kind of point x is (fixed? n-periodic? eventually n-periodic? none of these?).

2. Let $x \in K$ be the value with ternary expansion $0.\overline{0022}$. Give the ternary expansions of $T(x), T^2(x)$, and $T^3(x)$ and describe what kind of point x is (fixed? n-periodic? eventually n-periodic? none of these?).

3. Let $x \in K$ be the value with ternary expansion $0.\overline{002220}$. Give the ternary expansions of $T(x), T^2(x)$, and $T^3(x)$ and describe what kind of point x is (fixed? n-periodic? eventually n-periodic? none of these?).

4. In Desmos, I made a plot of $T^4(x)$ and found that it intersected with $y = x$ at various x-values (which are thus period-4 points). Of course, Desmos does not give the exact x-value of the intersection points, only decimal approximations. One of these was the value $x \approx 0.0366$. Find the exact value of this intersection point by trying to find its ternary expansion. *Hint: it will have a repeating ternary expansion with an 8-digit pattern that repeats.*

Problem 2. Our exercise in class and the problem above show that finding periodic points of T is not straightforward. Let's consider a new map $\sigma : K \to K$ that acts on ternary expansions in a simpler way: given any $x = 0.s_1s_2s_3\ldots$

$$\sigma(x) = 0.s_2s_3s_4\ldots.$$

Notice that σ just deletes the first coefficient in the ternary expansion of x like T did for values of $x \leq 1/3$ but σ acts this way for *every* element of K. This map is called the *shift map* and we will see that it's much easier to find fixed points for σ.

1. There are two fixed points of σ. State their ternary expansions.

2. There are two period-2 points of σ that are not fixed. State their ternary expansions.

3. There are six period-3 points of σ that are not fixed. State their ternary expansions.

4. In words, how would you identify a period-4 point of σ?

5. How many period-n points does σ have?