§6.1 More on bifurcations of Q_c

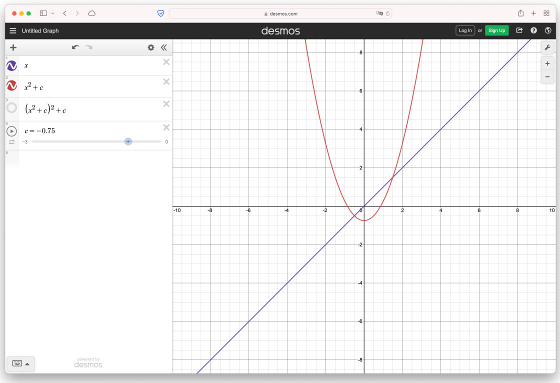
Today we're going to discuss what is called a period-doubling bifurcation.

This is the phenomenon of having new periodic points appear whose period is double the period of existing periodic points.

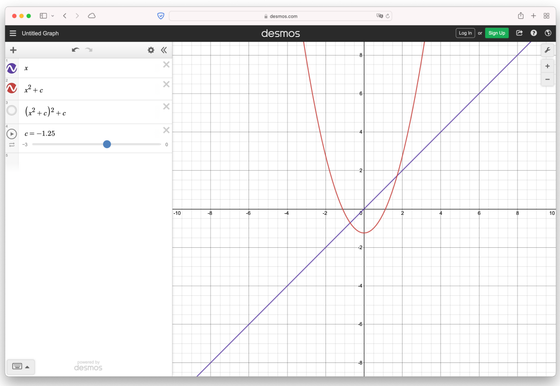
Example Can we graphically see period-2 points appear in the family $Q_c(x)$ in the way we saw fixed points appear when $Q_c(x) = x$ and $Q_c'(x) = 1$?

Let's use Desmos to investigate $Q_c^2(x)$.

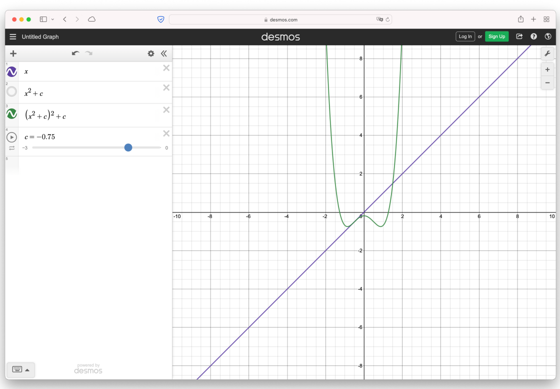
Geometry



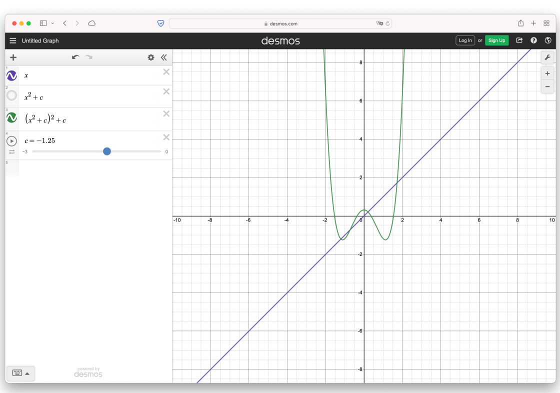
Notice at $c = -3/4$, fixed point p_- is neutral



At $c = -\frac{5}{4}$ nothing interesting seems to happen with $Q_c(x)$



At $c = -3/4$, $Q_c^2(x)$ is tangent to $y=x$ at $x=p_-$



Now that $c < -3/4$ period 2 points have appeared! At $c = -5/4$, they are neutral.

Algebra

Note $Q_c^2(x) = (x^2+c)^2 + c$

$$= x^4 + 2cx^2 + c^2 + c$$

and $Q_c^2(x) = x$ should have at most 4 solutions. We actually know 2 of them : p_+ and p_- since

$$Q_c^2(p_\pm) = Q_c(Q_c(p_\pm))$$

$$= Q_c(p_\pm) = p_\pm .$$

Therefore

$$0 = Q_c^2 - x = x^4 + 2cx^2 - x + c^2 + c$$

$$= (\text{some 2 degree factor})(x-p_+)(x-p_-)$$

We can do long division and find

$$\frac{x^4 + 2cx^2 - x + c^2 + c}{(x - p_+)(x - p_-)} = x^2 + x + c + 1$$

Questions

① What are the solutions to $Q_c^2(x) = x$?

② What are the period 2 points?

③ When are the period 2 points attracting / repelling / neutral?

What range of c values?

Q1

$$x^2 + x + c + 1 = 0$$

when $$x = \frac{-1 \pm \sqrt{1-4(c+1)}}{2}$$

This equation has solutions when

$$1 - 4(c+1) \geq 0$$

$$\Rightarrow \quad 4(c+1) \leq 1$$

$$\Rightarrow \quad c+1 \leq \frac{1}{4}$$

$$\Rightarrow \quad c \leq \frac{-3}{4}$$

When $c = \frac{-3}{4}$, $x = \frac{-1}{2} = p_-$ is a fixed point

When $c < \frac{-3}{4}$

$$q_- = \frac{-1 - \sqrt{1-4(c+1)}}{2} \quad \text{and} \quad q_+ = \frac{-1 + \sqrt{1-4(c+1)}}{2}$$

are the two solutions.

Summary

When $-\frac{3}{4} < c < \frac{1}{4}$ $Q_c^2(x) = x$

has 2 solutions, p_- and p_+, the

fixed points of Q_c

When $c = -\frac{3}{4}$, $Q_c^2(x) = x$ still has

only 2 solutions.

When $c < -\frac{3}{4}$, $Q_c^2(x) = x$ now has

4 solutions:

$$\underbrace{p_-, p_+}_{\text{fixed points}}, \underbrace{q_-, q_+}_{\text{period-2 points}}$$

Q2 When are they attracting / repelling / neutral ?

$$Q_c'(x) = 2x$$

$$\left|(Q_c^2)'(q_-)\right| = \left|Q_c'(q_-) \cdot Q_c'(q_+)\right|$$

$$= 4\left|q_- \cdot q_+\right|$$

$$= 4\left|\left(\frac{-1-\sqrt{1-4(c+1)}}{2}\right)\left(\frac{-1+\sqrt{1-4(c+1)}}{2}\right)\right|$$

$$= \left|\left(-1-\sqrt{1-4(c+1)}\right)\left(-1+\sqrt{1-4(c+1)}\right)\right|$$

$$= \left|1-(1-4(c+1))\right|$$

$$= |4(c+1)|$$

$$|4(c+1)| < 1$$

$$\iff -1 < 4(c+1) < 1$$

$$\iff -\frac{1}{4} < c+1 < \frac{1}{4}$$

$$\iff -\frac{5}{4} < c < -\frac{3}{4}$$

range of values when 2-cycle is attracting

$|4(c+1)| = 1$ when

$$\iff 4(c+1) = \pm 1$$

$$\iff c+1 = \pm\frac{1}{4}$$

$$\iff c = -1 \pm \frac{1}{4}$$

$$= -\frac{5}{4} \text{ or } \cancel{c = -\frac{3}{4}}$$

neutral

$|4(c+1)| > 1$ when

$$c < -\frac{5}{4}$$

repelling