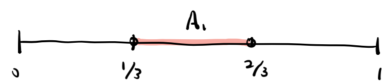
§7.3 The Cantor Middle-Thirds Set

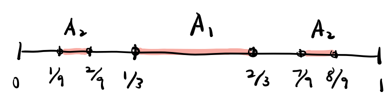
Here we construct a set that's similar to the set Λ of initial seeds of $Q_c(x)$ that don't escape to ∞.

An iterative construction Consider $I = [0,1]$

① Remove from I, the interval $A_1 = (1/3, 2/3)$



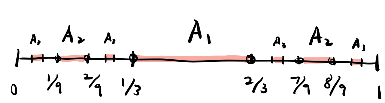
② $I - A_1$ has two intervals. Remove $A_2 = (1/9, 2/9) \cup (7/9, 8/9)$ the middle third of each



③ $I - (A_1 \cup A_2)$ has 4 intervals.

Remove $A_3 = (\frac{1}{27}, \frac{2}{27}) \cup (\frac{7}{27}, \frac{8}{27}) \cup (\frac{19}{27}, \frac{20}{27}) \cup (\frac{25}{27}, \frac{26}{27})$

the middle third of each



④ Continue removing middle thirds ad infinitum and let $K = I - \bigcup_{n=1}^{\infty} A_n$

This set is called the Cantor Middle-Thirds Set.

Facts: (1) The total length of what's removed from I to get K is 1.

(2) K has infinitely many elements; in fact it has uncountably many elements (this means we can make a one-to-one correspondence between K and $\mathbb{R}$)

Formula Recall from calculus that a geometric series $\sum_{n=0}^{\infty} ax^n$ converges when $|x| < 1$ and it sums to $\frac{}{1-x}$ when it converges.

Exercise Use this formula to prove fact 1.

Definition A real number $x \in [0,1]$ has **ternary expansion** $0.s_1 s_2 s_3 \ldots$ where s_n is either 0, 1, or 2 for each n if

$$x = \sum_{n=1}^{\infty} \frac{s_n}{3^n}$$

Example Find the real number represented by the ternary expansion $0.\overline{02} = 0.020202\ldots$

$$\frac{0}{3} + \frac{2}{3^2} + \frac{0}{3^3} + \frac{2}{3^4} + \cdots = 2\left(\frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + \cdots\right)$$
$$= 2\sum_{n=1}^{\infty}\left(\frac{1}{9}\right)^n = \frac{\frac{2}{9}}{1-\frac{1}{9}} = \frac{1}{4}$$

Exercise Can you find a ternary expansion of $1/3$? How about $8/9$?

Exercise Suppose $x \in [0, 1/3)$. What do you know must be true about its ternary expansion? What if $x \in \left(\frac{1}{3}, \frac{2}{3}\right)$ or $x \in \left(\frac{2}{3}, 1\right]$?

Exercise Suppose $x \in \left[0, \frac{1}{9}\right)$. What can you say about its ternary expansion?
What if $x \in \left(\frac{1}{9}, \frac{2}{9}\right)$ or $x \in \left(\frac{2}{9}, \frac{1}{3}\right)$?

To come. What we're doing is trying to build a proof that every element of K corresponds to a ternary expansion and this will lead us to making a correspondence between K and $[0,1]$.