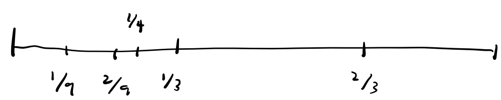
Finding ternary expansion of a number

Let $x \in [0,1]$. How do we find $0.s_1 s_2 s_3 \ldots$ such that $x = \sum_{i=1}^{\infty} \frac{s_i}{3^i}$?

Example Find ternary expansion of $x = \frac{1}{4}$.



Why must $s_1 = 0$?

If $s_1 = 1$, then $$\sum_{i=1}^{\infty} \frac{s_i}{3^i} = \frac{1}{3} + \frac{s_2}{3^2} + \frac{s_3}{3^3} + \ldots$$
$$\geq 1/3 > x$$

So there's no way the ternary expansion could equal x if $s_1 = 1$ or 2.

What about s_2 ?

If $s_2 = 1$, then $$\sum_{i=1}^{\infty} \frac{s_i}{3^i} = \frac{0}{3} + \frac{1}{3^2} + \frac{s_3}{3^3} + \ldots$$
$$= \frac{1}{9} + \frac{s_3}{3^3} + \frac{s_4}{3^4} + \ldots$$

Could this possibly sum to $\frac{1}{4}$?

The biggest value we could get is

$$\frac{1}{9} + \frac{2}{3^3} + \frac{2}{3^4} + \frac{2}{3^5} + \ldots$$

$$= \frac{1}{9} + \frac{a}{1-x} \quad \text{where} \quad a = \frac{2}{3^3}, \ x = \frac{1}{3}$$

$$= \frac{1}{9} + \frac{2/27}{1-1/3} = \frac{1}{9} + \frac{2/27}{2/3} = \frac{1}{9} + \frac{1}{9} = \frac{2}{9} < x$$

So we couldn't possibly sum to x.

We need $s_2 = 2$.

So far:

$$\frac{1}{4} = 0.02 s_3 s_4 s_5 \ldots$$

To find s_3 we need to decide which interval,

$\left(\frac{2}{9}, \frac{2}{9} + \frac{1}{27}\right)$ or $\left(\frac{2}{9} + \frac{1}{27}, \frac{2}{9} + \frac{2}{27}\right)$ or $\left(\frac{2}{9} + \frac{2}{27}, \frac{1}{3}\right)$,

that $1/4$ falls inside of.

(Equivalently we can compute

$3^3\left(\frac{1}{4} - \frac{0}{3} - \frac{2}{3^2}\right)$ and see if we

get a number in $(0,1)$, $(1,2)$, or $(2,3)$)

We'll find that it's in the first interval,

so $s_3 = 0$. We'll also find $s_4 = 2$,

and $x = 0.0202 s_5 s_6 \ldots$

In fact $x = 0.\overline{02}$.

<u>Question</u> Is $1/4$ in the Cantor middle thirds set?

<u>Answer</u> Yes! Notice its ternary expansion has only 0's and 2's; it never falls in a "middle third" interval.

<u>Lemma</u> If $x \in K$, its ternary expansion has only 0's and 2's (no 1's).

<u>Lemma</u> Every number in $[0,1]$ has binary expansion $0.b_1 b_2 b_3 \ldots$ where each $b_1, b_2, b_3, \ldots$ is either 0 or 1.

Theorem The Cantor Middle Thirds set K is an uncountably infinite set (meaning there is a correspondence between K and $[0,1]$).

Proof We claim that given any number in $[0,1]$, it can be matched with an element of K. Let $x \in [0,1]$. Then it has a binary expansion $0.b_1 b_2 b_3 \ldots$ The b_i's that are 1 can be changed to 2 and we get a ternary expansion of 0's and 2's. This must be an element of K. So given <u>any</u> element of $[0,1]$ we get an element of K, so K must have at least as many elements as $[0,1]$, which is uncountably infinite.