§ 10.1 Definition of Chaos

We've spent some time focusing on examples (Q_c, L_λ, T) and their orbits that stay bounded. We claim these orbits have "chaotic" behavior but what does that mean?

Devaney's definition of chaos

A dynamical system $F: X \to X$ is chaotic if

1. the periodic points of F are dense in X
2. F is transitive
3. F exhibits sensitive dependence on initial conditions.

There are some things to unpack here.

We'll give formal definitions but what matters first is intuitive understanding.

Def's

1. A set $A \subseteq X$ is dense if for any $x \in X$, we can make a sequence (a_n) of elements of A that converges to x

   (Intuition: points of A can get arbitrarily close to points of X)

2. F is transitive if for any $x, y \in X$ and any $\varepsilon > 0$, there exists a point $z \in X$ such that ① z is within ε-units of x and ② the orbit of z gets within ε-units of y

   (Intuition: orbits of F explore and get close to all the points of X)

(3) F has <u>sensitive dependence on initial conditions</u> if there exists $\beta > 0$ such that for any $x \in X$ and any $\varepsilon > 0$, there is a $y \in X$ and a $k \geq 1$ such that x, y are within ε-units of each other and $F^k(x)$ and $F^k(y)$ are at least β units apart.

(<u>Intuition</u>: we can always find orbits whose initial seeds are arbitrarily close but whose orbits eventually drift apart)