Announcements

- HW0 due tomorrow on Gradescope at 5pm
  - correction to Problem 7

    $x_0 = 3$ should instead be $x_0 = 0.3$

    (the pdf is now updated)

- HW1 posted on class web page
  - due Wednesday at 5 pm
  - covers material from Week 1.

Last time we thought about

$$F(x) = 2x(1-x)$$

and we thought about taking an initial seed x_0 and getting the orbit of x_0 through iteration of F:

$$x_1 = F(x_0)$$

$$x_2 = F(x_1) = F^2(x_0)$$

$$\vdots$$

Def A fixed point of a map F is a value x_0 that satisfies equation $F(x) = x$.

Example Let's find the fixed points of some maps

① $F(x) = 2x(1-x)$

② $F(x) = x^2$

③ $F(x) = x^3$

④ $F(x) = \cos x$

either using algebra or at least through a picture/graph.

① Solve $2x(1-x) = x$

$$2x - 2x^2 = x$$

$$x - 2x^2 = 0$$

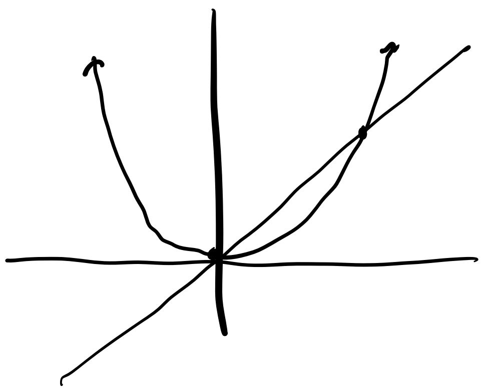
$$x(1-2x) = 0$$

$$x = 0 \quad \text{or} \quad 1 - 2x = 0$$

$$x = \tfrac{1}{2}$$

(2) What does it mean to solve $x = F(x)$ visually?

We find intersection between $y = x$ and $y = F(x)$



Algebraically, we solve

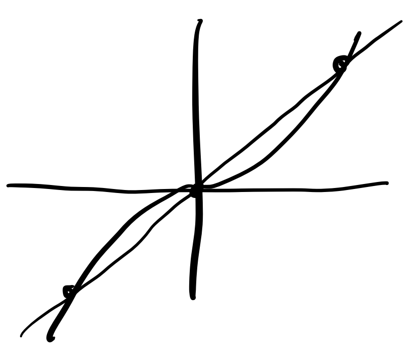
$$x^2 = x$$

$$\Rightarrow x^2 - x = 0$$

$$\Rightarrow x(x-1) = 0$$

$$\Rightarrow x = 0,\ x = 1$$

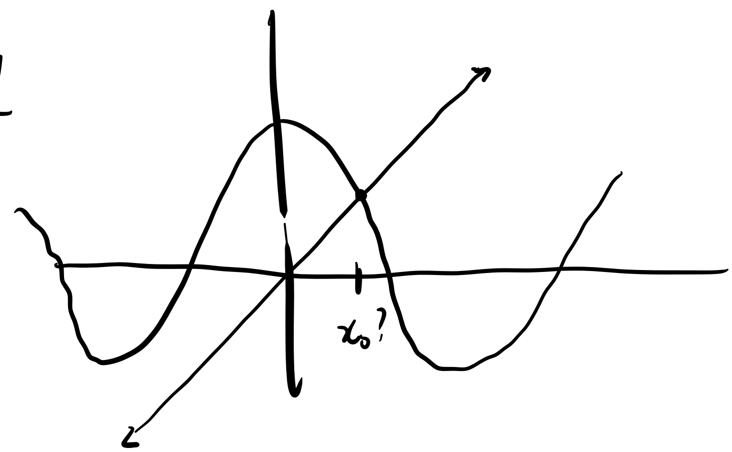
(3) $F(x) = x^3$



(4) $F(x) = \cos x$

<u>Algebra</u> $\cos x = x$ (intractable by hand)

<u>Usually</u>



Can we find the fixed point

by starting with an initial seed

and repeatedly apply the map $F(x) = \cos x$?

will we run into issues with

our fixed point not being converged

to? Does our initial seed matter?

Does the oscillating nature of $\cos x$ matter?

Example Consider $F(x) = x^2 - 1$.

Use initial seeds $x_0 = 0$, $x_0 = \sqrt{2}$

and find their orbits.

$x_0 = 0$

$$x_1 = F(x_0) = F(0) = 0^2 - 1 = -1$$

$$x_2 = F(x_1) = F(-1) = (-1)^2 - 1 = 1 - 1 = 0$$

$$x_3 = F(x_2) = F(0) = -1$$

This point $x_0 = 0$ is called a periodic point with period 2.

And the orbit $0, -1, 0, -1, \ldots$ is called a 2-cycle.

-1 is also a period 2 point.

$x_0 = \sqrt{2}$

The orbit here is

$$\sqrt{2},\ 1,\ 0,\ -1,\ 0,\ -1,\ 0,\ -1,\ \ldots$$

we call $x_0 = \sqrt{2}$ an eventually periodic point.

How do we find a period 2 point algebraically?

Solve the equation

$$F^2(x) = x$$

In this example

$$F^2(x) = F(F(x))$$
$$= F(x^2-1) = (x^2-1)^2 - 1$$

This is a tough algebraic equation to solve.

Problem 1. Suppose $a = 1.5$. Choose a few initial seeds $x_0 \in [0, 1]$ (for concreteness, let's say choose 5 different values) and use the `Iterator.m` and `time_series.m` MATLAB scripts to find the first 30 iterates of each of your initial seeds.

1. Record the results in two ways:

   (a) Make a 5 row, 2 column table listing the initial seed x_0 together with the value $F^{30}(x_0)$.

   (b) Save the image produced by the `time_series.m` script.

2. Given your data, can you make any conjectures about the behavior of orbits in general? Do all orbits of F (or almost all) behave in the same way? Does F have any fixed points? How can you interpret these fixed points in the context of the logistic map as a population model. Write a few sentences describing what you've seen in your data, pointing out any special orbits.

3. Find the two fixed points of F using algebra.

4. Later we will learn the terms *attracting fixed point* and *repelling fixed point.* In the context of what you've seen in your data, what do you think is meant by these terms? How would you classify the two fixed points you found in part 3? Write a sentence or two in response to these questions.

5. Which kind of fixed points (attracting or repelling) can be found (or at least approximated) using numerical simulation instead of having to do possibly complicated algebra? Which cannot? Why?

Problem 2. I don't want to give the impression that everything works out so nicely for every value of a. Try making similar time series plots for $a = 3.2$ and $a = 3.5$ and $a = 3.55$ with the seed value $x_0 = 0.5$ for each (I've chosen this seed value arbitrarily, but it's representative of what you'd see in general). For each value of a respond to the following questions.

1. Save a picture of your time series plot.
2. Do you believe F has any periodic points? What period? What is it about your time series plots that leads you to these conjectures?
3. What equation would you need to solve to prove the existence of periodic points?