Dynamics of the shift map

Let $\Sigma = \{ (s_1, s_2, \dots) : s_i = 0 \text{ or } 1 \}$

be the set of binary sequences and

let $\sigma : \Sigma \to \Sigma$ be the shift map:

$$\sigma(s_1, s_2, s_3, \dots) = (s_2, s_3, \dots).$$

It's possible to measure the distance between two binary sequences:

$$d\big((s_1, s_2, \dots), (t_1, t_2, \dots)\big)$$

$$= \frac{s_1 - t_1}{2} + \frac{s_2 - t_2}{2^2} + \frac{s_3 - t_3}{2^3} + \dots$$

We can also discuss <u>periodic</u> points of σ:

- $(0, 0, \ldots)$ and $(1, 1, 1, \ldots)$ are fixed points

- $(0, 0, 0, \ldots)$, $(1, 1, 1, \ldots)$, $(1, 0, 1, 0, 0, \ldots)$ and $(0, 1, 0, 1, \ldots)$ are period-2 points

- period-n points can be found by listing all the sequence that have a repeating pattern of length n

<u>Fact</u> The periodic points of σ are dense in Σ

<u>Idea of proof</u> Given any sequence $x = (s_1, s_2, \ldots)$ and any ε, find a value of n so that $\frac{1}{2^n} < \varepsilon$.

Let y be the period-n point that repeats $s_1, s_2, \ldots, s_n$, the first n components of x.

It can be shown $d(x, y) < \varepsilon$.

Fact The shift map σ is transitive.

Idea of proof

Let's make a very special sequence:

$$z = \Big(\underbrace{0,1}_{\substack{\text{possible}\\ \text{pattern of}\\ \text{length 1}}},\ \underbrace{\underbrace{0,0},\underbrace{0,1},\underbrace{1,0},\underbrace{1,1}}_{\substack{\text{possible patterns}\\ \text{of length 2}}},\ \underbrace{\ \dots\ }_{\substack{\text{possible patterns}\\ \text{of length 3}}}$$

that consists of possible patterns of length n for every n.

It's possible to show that for any $x, y \in \Sigma$ and any $\varepsilon > 0$ that

$$d(x, \sigma^m(z)) < \varepsilon \quad \text{for some } m$$

$$\text{and} \quad d(y, \sigma^p(z)) < \varepsilon \quad \text{for some } p$$

In other words the orbit of z gets arbitrarily close to any $x, y \in \Sigma$.

Itineraries

Consider the tent map $T : K \to K$, $T(x) = \begin{cases} 3x, & x \leq \frac{1}{2} \\ 3 - 3x & x > \frac{1}{2} \end{cases}$

Let $I_0 = [0, \frac{1}{3}]$ and $I_1 = [\frac{2}{3}, 1]$

and let $x_0 \in K$ be arbitrary. We know

$$x_0 \in I_0 \quad \text{or} \quad x_0 \in I_1$$

$$T(x_0) \in I_0 \quad \text{or} \quad T(x_0) = I_1$$

$$\vdots \qquad\qquad \vdots$$

Given $x_0 \in K$, define $S(x_0) \in \Sigma$ be the binary sequence $S(x_0) = (s_0, s_1, s_2, \dots)$

where
$$s_n = 0 \quad \text{if} \quad T^n(x_0) \in I_0$$
$$s_n = 1 \quad \text{if} \quad T^n(x_0) \in I_1$$

$S(x_0)$ is called the <u>itinerary</u> of x_0

It describes the orbit of x_0 under T and it gives a way of "rethinking" the dynamics of T using the shift map.

$$\begin{tikzcd}
K \arrow[r, "T"] \arrow[d, "S"'] & K \arrow[d, "S"] \\
\Sigma \arrow[r, "\sigma"'] & \Sigma
\end{tikzcd}$$

<u>Fact</u> $T(x) = S^{-1} \circ \sigma \circ S(x)$

and the tent map is chaotic because σ is chaotic.

<u>Idea</u> All the dynamical properties (eg. periodic points) of σ can be converted to ones of T.

For example if p is a fixed of σ, $S^{-1}(p)$ is a fixed point of T:

$$\begin{aligned}
T(S^{-1}(p)) &= (S^{-1} \circ \sigma \circ S)(S^{-1}(p)) \\
&= S^{-1} \circ \sigma \left(S(S^{-1}(p)) \right) \\
&= S^{-1} \circ \sigma(p) \\
&= S^{-1}(p)
\end{aligned}$$