Tuesday worksheet Problem 1

$$F(x) = x^2 - \frac{x}{2}$$

Fixed points

$F(x) = x$

$\Rightarrow \quad x^2 - \frac{x}{2} = x$

$\Rightarrow \quad x^2 - \frac{3}{2}x = 0$

$\Rightarrow \quad x\left(x - \frac{3}{2}\right) = 0$

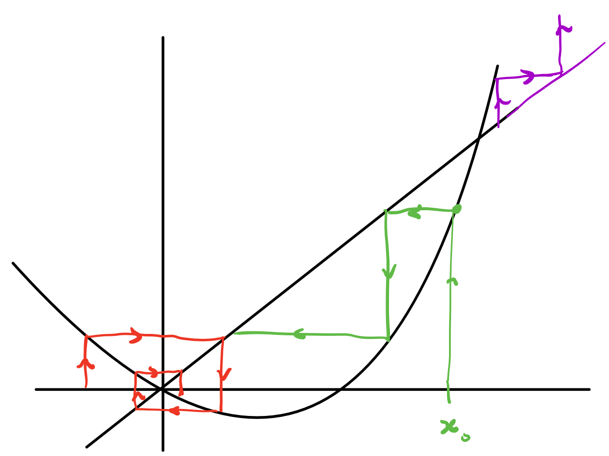
$\Rightarrow \quad x = 0, \ \frac{3}{2}$

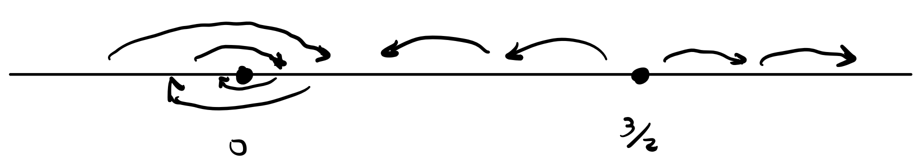
$F'(x)$ at fixed points

$F'(x) = 2x - \frac{1}{2}$

$F'(0) = -\frac{1}{2}$

$F'\left(\frac{3}{2}\right) = 2.5$





Orbits repel from $\frac{3}{2}$, attract toward 0, spiraling inward.

$$F(x) = 2.5x - x^2$$

<u>Fixed points</u>

$F(x) = x$

$\Rightarrow \quad 2.5x - x^2 = x$

$\Rightarrow \quad 1.5x - x^2 = 0$

$\Rightarrow \quad x(1.5 - x) = 0$

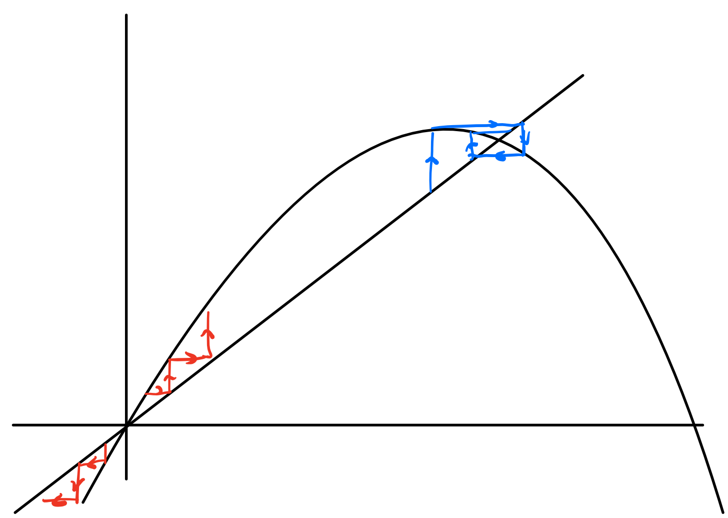
$\Rightarrow \quad x = 0, 1.5$

<u>$F'(x)$ at fixed points</u>

$F'(x) = 2.5 - 2x$

$F'(0) = 2.5$

$F'(1.5) = -0.5$



Orbits repel from 0, attract to 2.5, spiraling inward to 2.5.

§5.4 More on attraction, repulsion.

Last time We showed when $0 < F'(p) < 1$ for a fixed point p, orbits that start nearby must converge (though we didn't prove they converge to p.) Can we handle negative slopes too?

Def Let p be a fixed point of F.

- If $|F'(p)| < 1$, p is called an attracting fixed point.
- If $|F'(p)| > 1$, p is called a repelling fixed point.
- If $|F'(p)| = 1$, p is called a neutral fixed point (it might "attract" or "repel" or both!)

Questions Suppose $|F'(p)| < 1$.

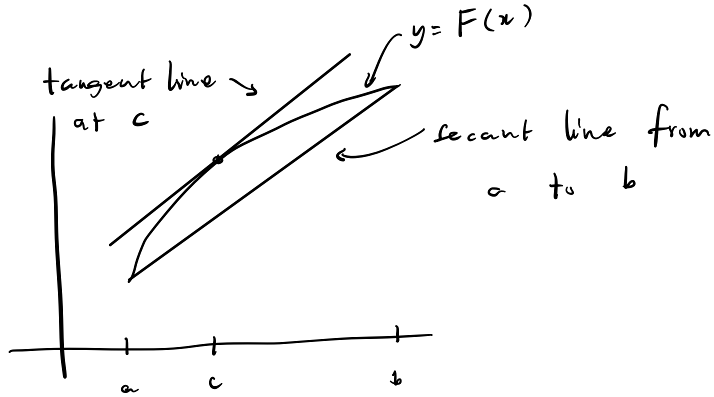
① How can we prove convergence <u>to p</u>?

② Can we even discuss <u>speed</u> of convergence to p?

Mean Value Theorem If F is differentiable on the interval (a,b), there exists a value $c \in (a,b)$ such that

$$F'(c) = \frac{F(b) - F(a)}{b - a}$$

"there's a place where tangent slope equals secant slope"



Notice,

since $|F'(p)| < 1$, there is a value

$\lambda < 1$ such that

① $|F'(p)| < \lambda$

② $|F'(x)| < \lambda$ for all x that

are nearby p, ie in an interval

around p.

Next,

let x_0 be an initial seed near p.

Notice
$$
\begin{aligned}
|F(x_0) - F(p)| &= |F'(c_0)(x_0 - p)| \quad \text{by MVT} \\
&= |F'(c_0)|\,|x_0 - p| \\
&< \lambda |x_0 - p|
\end{aligned}
$$

for some c_0 between x_0 and p.

Further, $$|F(x_1)-F(p)| = |F'(c_1)(x_1-p)| \quad \text{by MVT}$$
$$= |F'(c_1)||x_1-p|$$
$$< \lambda |x_1-p|$$
$$= \lambda |F(x_0)-F(p)|$$
$$< \lambda (\lambda |x_0-p|)$$
$$= \lambda^2 |x_0-p|$$
for some c_1 between x_1 and p.

And $$|F(x_2)-F(p)| = |F'(c_2)(x_2-p)|$$
$$= |F'(c_2)||x_2-p|$$
$$< \lambda |x_2-p|$$
$$= \lambda |F(x_1)-F(p)|$$
$$< \lambda (\lambda^2 |x_0-p|)$$
$$= \lambda^3 |x_0-p|.$$

In *general*, we see

$$|x_n - p| = |F(x_{n-1}) - F(p)|$$
$$< \lambda^n |x_0 - p|.$$

What happens when $n \to \infty$ on the right side?

It goes to 0 since $\lambda < 1$!

This is telling us the distance from x_n to p is decreasing to 0 exponentially fast. So we've proven convergence to the fixed point and we even understand the "speed" of convergence!

Here's what we've just proved:

<u>Theorem</u> Suppose p is an attracting fixed point of F (meaning $|F'(p)| < 1$).

Then there is an interval I that contains p and in which the following is satisfied: for any initial seed $x_0 \in I$, $F^n(x_0) \in I$ for all n and $F^n(x_0) \to p$ as $n \to \infty$.