Discussion of Repelling Fixed Point Theorem

Theorem Suppose p is repelling fixed point of F (meaning $|F'(p)|>1$). Then there is an interval I that contains p in its interior and which satisfies the following: if $x_0 \in I$ and $x_0 \neq p$, then there is an integer n so that $F^n(x_0) \notin I$.

Remarks

- $F'(x)$ is assumed to be continuous. This means $|F'(p)|>1$ and $|F'(x)|>1$ for all x that are "near" p. In other words $|F'(x)|$ can't get too small.

- There is a value $\lambda > 1$ such that
    - $|F'(p)| > \lambda$
    - $|F'(x)| > \lambda$ for all x near p

We let I be the interval of x-values around p that satisfy $|F'(x)| > \lambda$.

- Using arguments like we made last time we can show

for any $x_0 \in I$,

$$\underbrace{|F^n(x_0) - p|}_{\text{distance from } F^n(x_0) \text{ to } p} > \lambda^n |x_0 - p|$$

- Notice that we can find integer n that makes λ^n as large as we'd like.

§ 5.4 Neutral fixed points

Definition Recall that if p is a fixed point of F and $|F'(p)| = 1$, we call p a neutral fixed point.

If orbits that start near p converge to p, we call p a weakly attracting fixed point.

If the repel away, we call p weakly repelling.

It's possible that orbits starting on one side converge to but repel away when starting from the other.

Examples

① $F(x) = x - x^2$

fixed points

$F(x) = x$

$\Rightarrow \quad x - x^2 = x$

$\Rightarrow \quad x^2 = 0$

$\Rightarrow \quad x = 0$

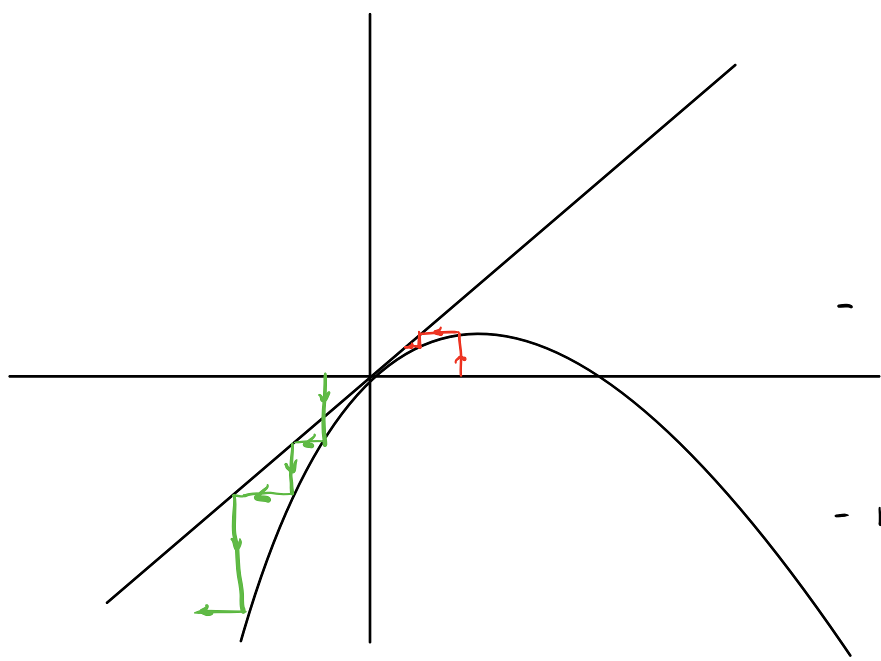
Derivatives at fixed point

$F'(x) = 1 - 2x$

$F'(0) = 1$ (so 0 is neutral)

$F''(x) = -2$

$F''(0) = -2$ (F is concave down at 0)



Summary

- when $x_0 > 0$

  $F^n(x_0) \to 0$

- when $x_0 < 0$

  $F^n(x_0) \to -\infty$

(2) $F(x) = x - x^3$

<u>fixed points</u>

$F(x) = x$

$\Rightarrow x - x^3 = x$

$\Rightarrow \quad x^3 = 0$

$\Rightarrow \quad x = 0$

<u>Derivatives at fixed points</u>

$F'(x) = 1 - 3x^2$

$F'(0) = 1$ ← neutral fixed point

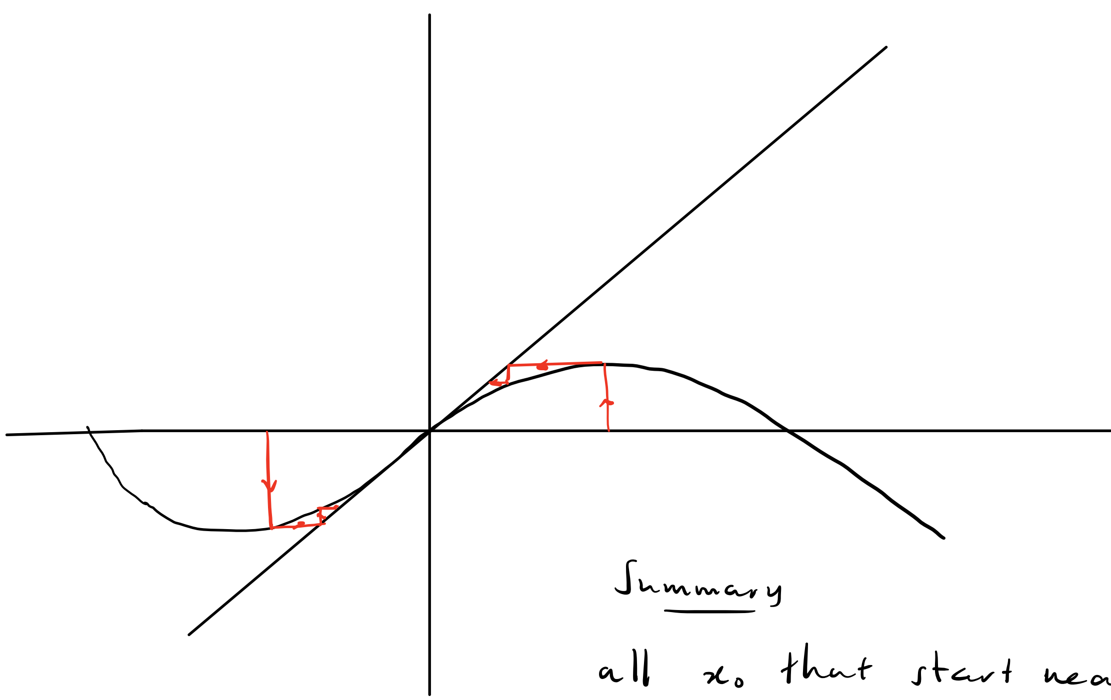
$F''(x) = -6x$

$F''(0) = 0$

$F'''(x) = -6$

$F'''(0) = -6$

F has inflection point at 0, concavity changes from concave up to concave down



<u>Summary</u>

all x_0 that start near 0 converge to 0.

Worksheet Problem 1

$F(x) = x^3 + x$

Fixed points

$x^3 + x = x$

$x^3 = 0$

$x = 0$

Derivatives

$F'(x) = 3x^2 + 1$

$F'(0) = 1$ (so 0 is neutral)

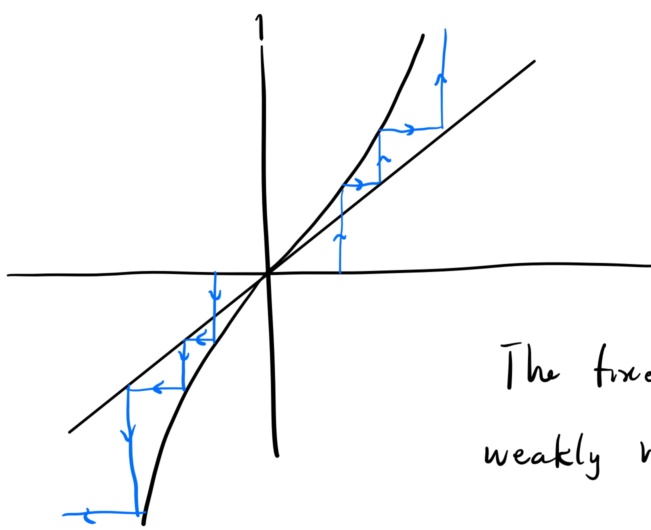
$F''(x) = 6x$

$F''(0) = 0$

$F'''(x) = 6$

$F'''(0) = 6 > 0$

inflection point at 0, and concavity changes from ⌢ to ⌣



The fixed point is weakly repelling!

$F(x) = x^2 - x$

Fixed points

$x^2 - x = x$

$x^2 - 2x = 0$

$x = 0, 2$

Derivatives

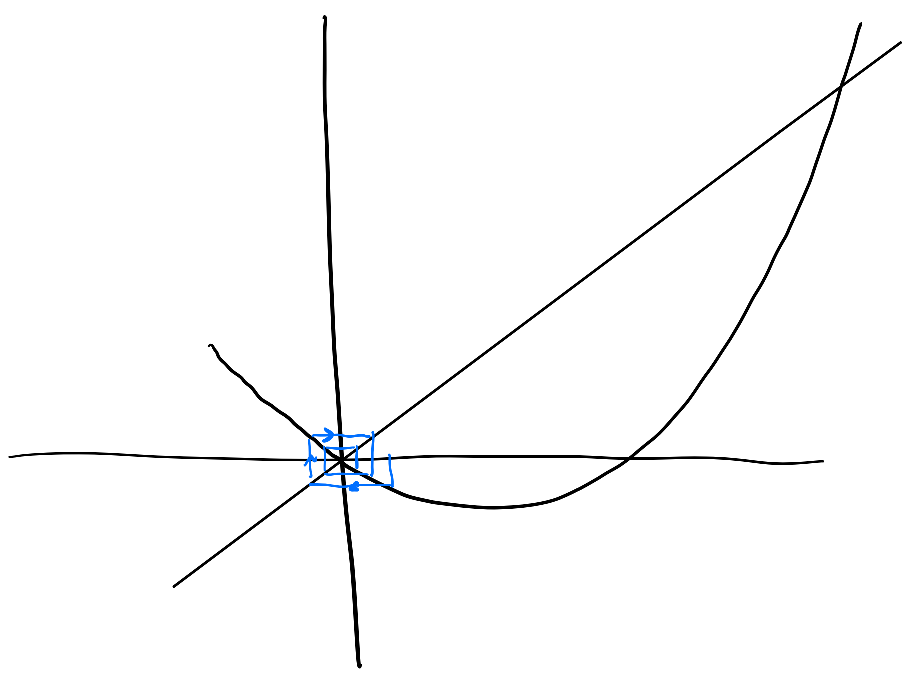
$F'(x) = 2x - 1$

$F'(0) = -1$, $F'(2) = 3$

(neutral) (repelling)

$F''(x) = 2$

$F''(0) = 2$ ← concave up



This one is tricky but MATLAB experimenting shows 0 is weakly attracting (orbits spiral in).