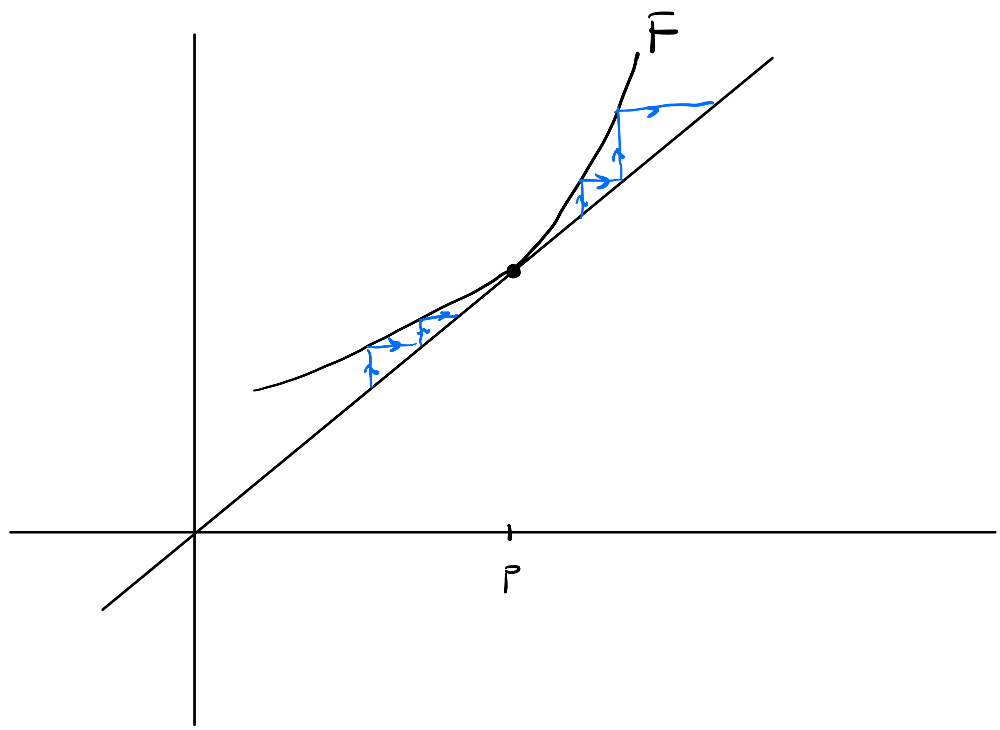
Discussion of Worksheet Problem 2

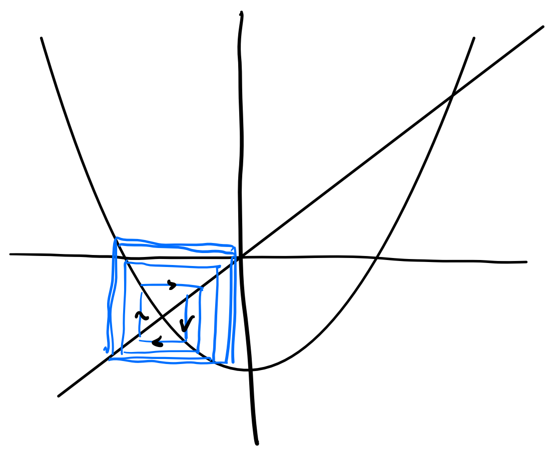
Suppose F has a neutral fixed point at with $F'(p) = 1$ and $F''(p) > 0$.



F must be tangent to $y = x$ at p and concave up. This gives weak attraction from the <u>left</u> but weak repulsion from the <u>right</u> of p.

§ 5.5 Periodic points

Let's consider $F(x) = x^2 - 1$. It has 2 fixed points $\left(\begin{array}{l} x^2 - 1 = x \\ \Rightarrow x^2 - x - 1 = 0 \Rightarrow x = \frac{1 \pm \sqrt{5}}{2} \end{array} \right)$



But it also seems to have period 2 points that get attracted to when we start an orbit near one of the fixed points. Let's talk about attraction and repulsion to cycles.

Consider $F^2(x) = F(F(x))$

$$= (x^2-1)^2 - 1$$

$$= x^4 - 2x^2$$

What are its fixed points?

$$x^4 - 2x^2 = x$$

$$x^4 - 2x^2 - x = 0$$

$$x(x^3 - 2x - 1) = 0$$

$$x(x+1)(x^2-x-1) = 0$$

$$x = 0, -1, \frac{1 \pm \sqrt{5}}{2}$$

<u>Check</u> $\left.\begin{array}{l} F(0) = 0^2 - 1 = -1 \\ F(-1) = (-1)^2 - 1 = 0 \end{array}\right\}$ so $0, -1$ form a 2-cycle

<u>Question</u> if we iterate F^2, do orbits attract or repel when they start near 0 and -1?

$(F^2)'(x) = 4x^3 - 4x$, so

$(F^2)'(0) = 0 < 1$ and $(F^2)'(-1) = 0 < 1$

Answer: 0 and -1 are attracting fixed points of F^2. This means when we do 2 iterates of F, we'll begin to attract to 0 when we start near 0 and attract to -1 when we start near -1.

Definition: Let x_0 be a periodic point of period n. Then it is

- attracting if $|(F^n)'(x_0)| < 1$
- repelling if $|(F^n)'(x_0)| > 1$
- neutral if $|(F^n)'(x_0)| = 1$

Example Let $F(x) = -\frac{3}{2}x^2 + \frac{5}{2}x + 1$.

We're told 0 is a period 3 point.

$x_0 = 0$, $\quad x_3 = F(2) = 0$,

$x_1 = F(0) = 1$, $\quad x_4 = 1$,

$x_2 = F(1) = 2$, $\quad x_5 = 2, \ldots$

Is 0 attracting, repelling, or neutral?

What about 1 and 2?

Chain rule How do we differentiate $f(g(x))$?

$$\left(f(g(x))\right)' = f'(g(x)) \cdot g'(x)$$

How about $F^2(x)$?

$$\left(F(F(x))\right)' = F'(F(x)) \cdot F'(x)$$

How about $F^3(x)$?

$$\left(F(F^2(x))\right)' = F'(F^2(x)) \cdot \left(F^2(x)\right)'$$

$$= F'(F^2(x)) \cdot F'(F(x)) \cdot F'(x)$$

Conclusion $(F^3)'(x_0)$

$$= F'(F^2(x_0)) \cdot F'(F(x_0)) \cdot F'(x_0)$$

$$= F'(x_2) \cdot F'(x_1) \cdot F'(x_0)$$

Therefore

$$(F^3)'(0) = F'(2) \cdot F'(1) \cdot F'(0)$$

We only need to compute $F'(x)$ and evaluate it at points along the cycle!

$$F'(x) = -3x + \frac{5}{2}$$

$$F'(0) = \frac{5}{2}, \quad F'(1) = -\frac{1}{2}, \quad F'(2) = -\frac{7}{2}$$

So $(F^3)'(0) = \left(\frac{5}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-7}{2}\right) = \frac{35}{8} > 1$ (repelling)