Chaos and Feigenbaum's Constant

Dynamical Systems

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1.

The orbit diagram

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Orbit diagram: the dynamics of Qc for many different c values in one picture(an attempt)

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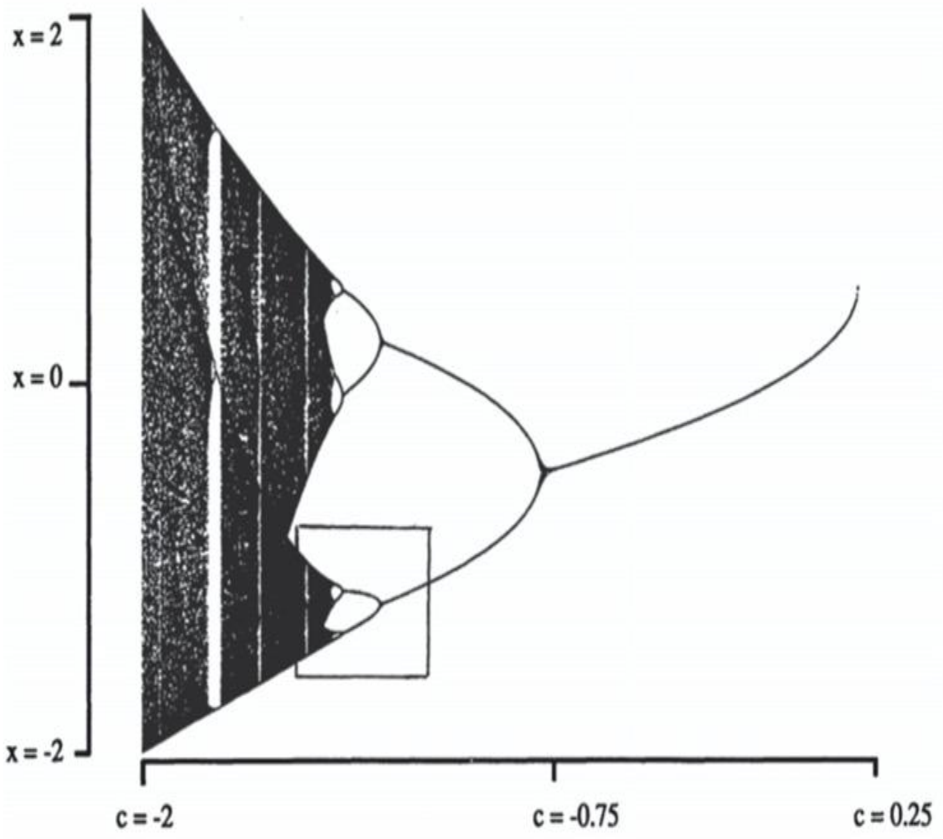
In the orbit diagram we plot the parameter *c* on the horizontal axis versus the *asymptotic orbit* of 0 under *Qc* on the vertical axis. We use the orbit of the critical point(0) to plot the orbit diagram.


x = 2
x = 0
x = -2
c = -2
c = -0.75
c = 0.25

Definition

Suppose $F : \mathbb{R} \longrightarrow \mathbb{R}$. A point x_0 is a *critical point* of F if $F'(x_0) = 0$.

(0 is the only critical point of Q_c)

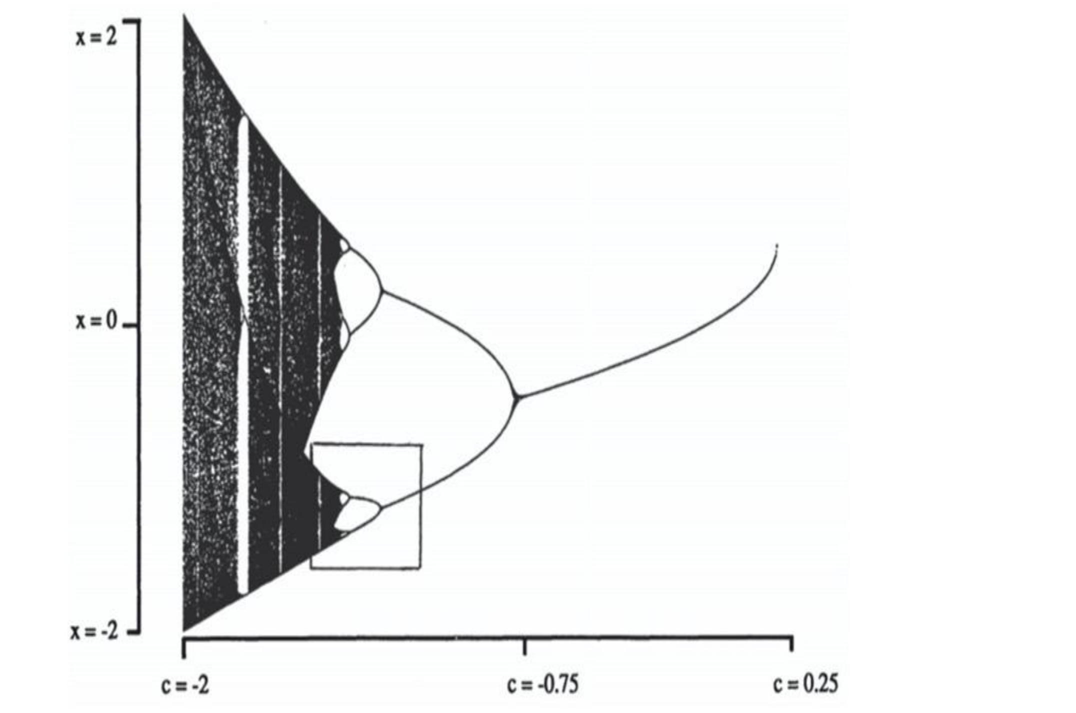
Observation 1

As c decreases, we seem to see a succession of period-doubling bifurcations. It seems that periodic points first appear in the order

1, 2, 4, 8,..., 2^n

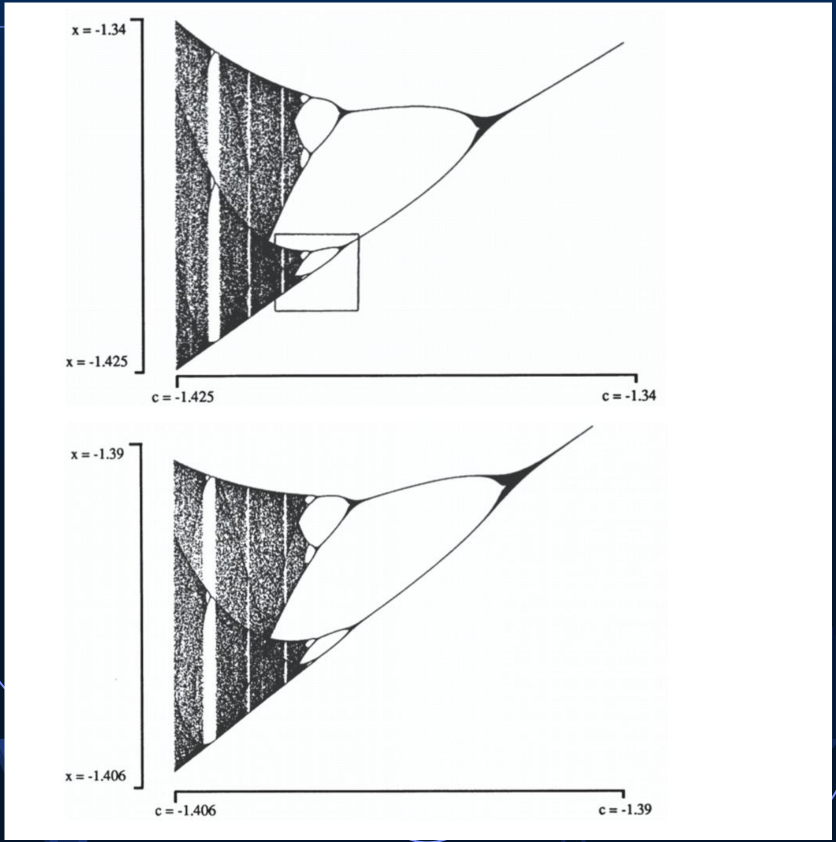
Observation 2

In each period-n window, we seem to see the appearance of an attracting n-cycle followed by a succession of period-doubling bifurcations.



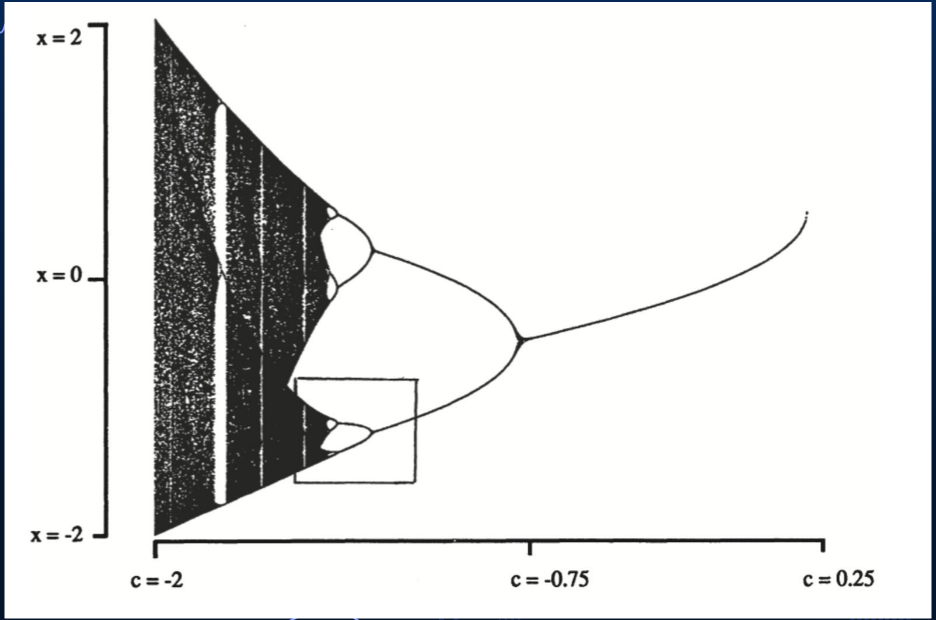
Observation 3

The orbit diagram appears to be self-similar: when we magnify certain portions of the picture, the resulting image bears a striking resemblance to the original figure.



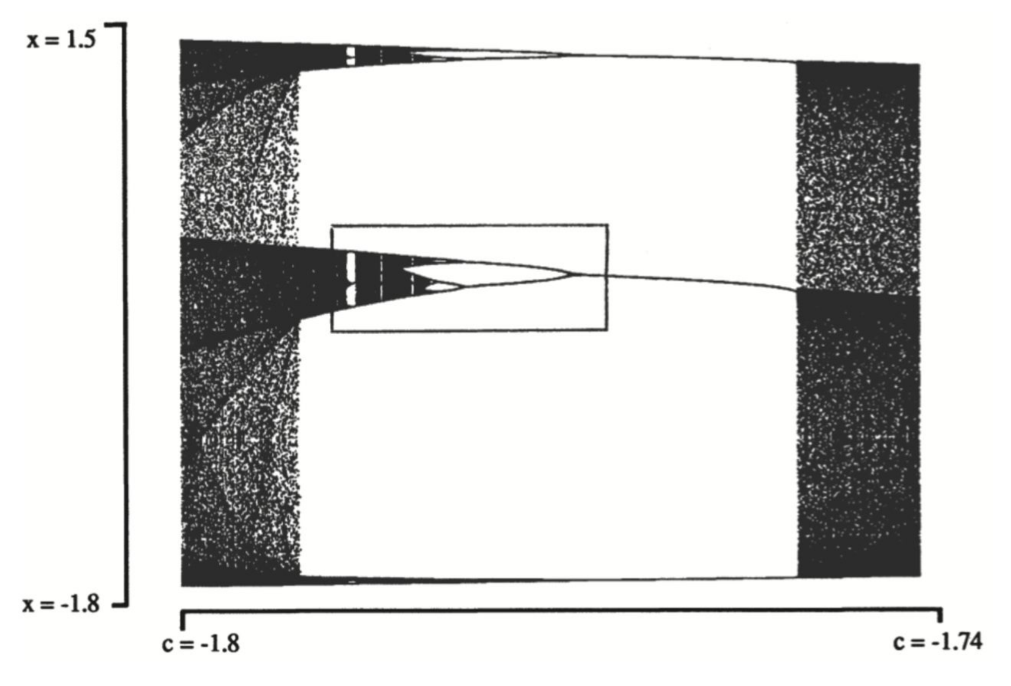
Observation 4

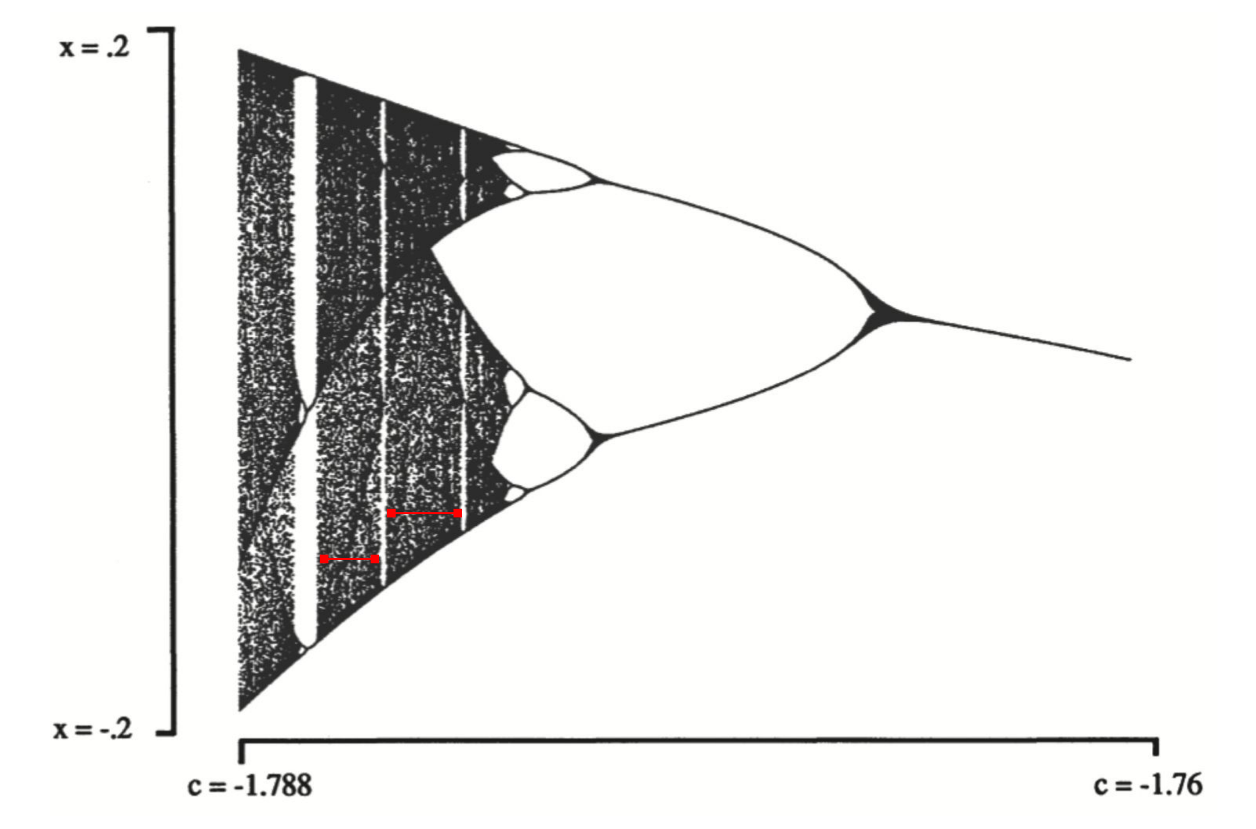
It appears there is at most one attracting cycle for each Q_c.



Observation 5

It appears that there is a large set of c-values for which the orbit of 0 is not attracted to an attracting cycle.





*This is a glimpse of **chaotic behavior.***

2.

The Period Doubling Route to Chaos

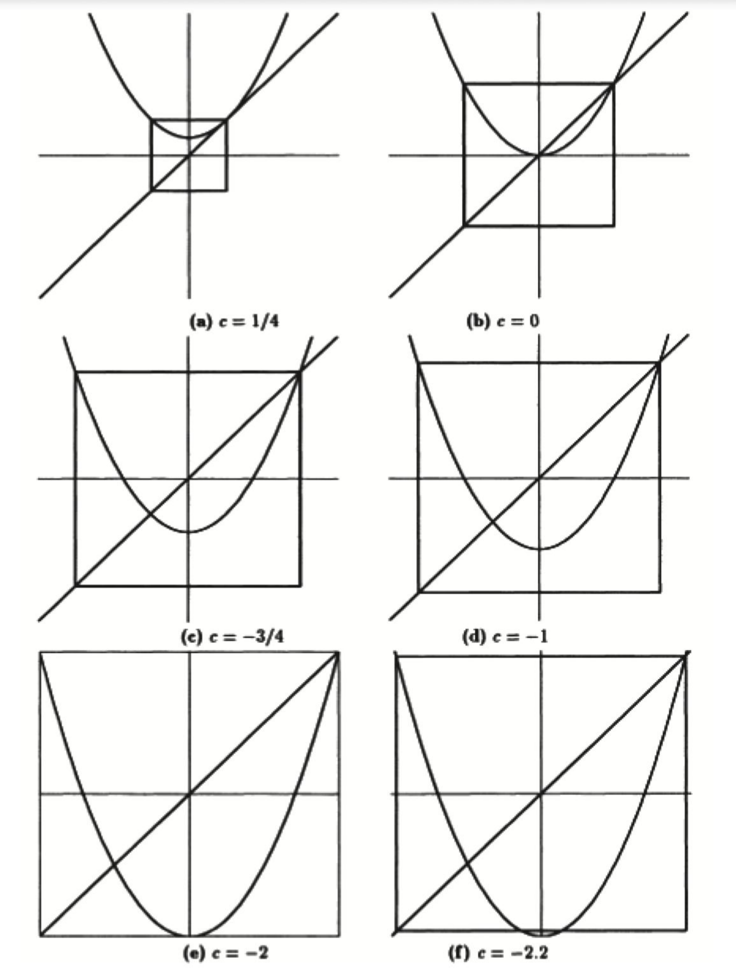


Figure : Graphs of Q_c

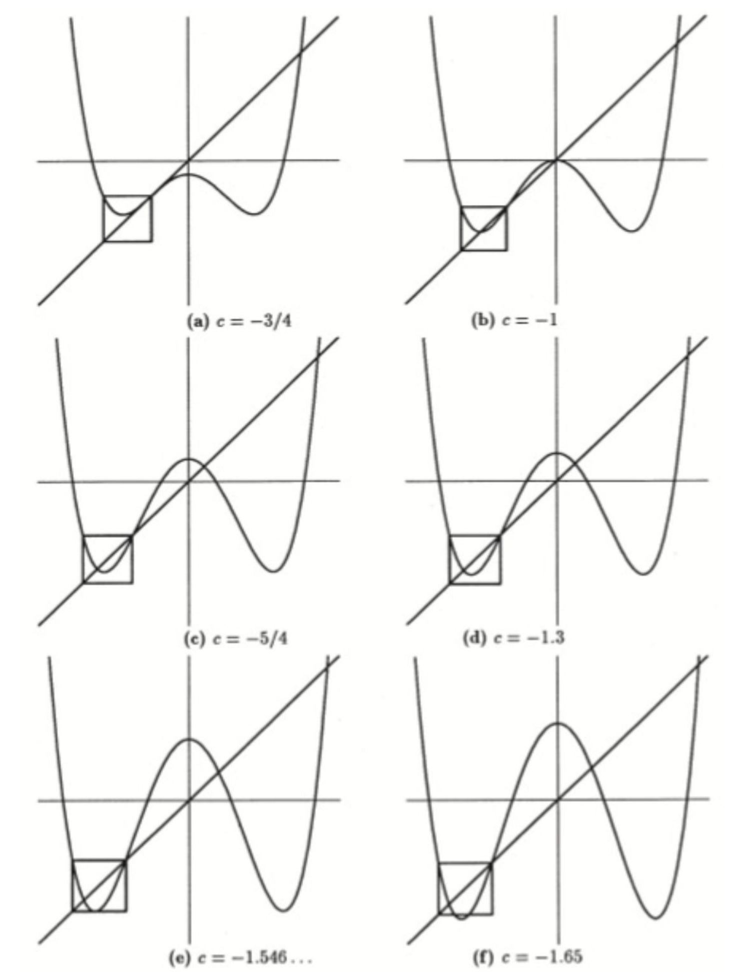


Figure : Graphs of Q^2_c

We can see that the graphs of Q^2_C resembles very closely to the corresponding graph of Q_C only on a much smaller interval

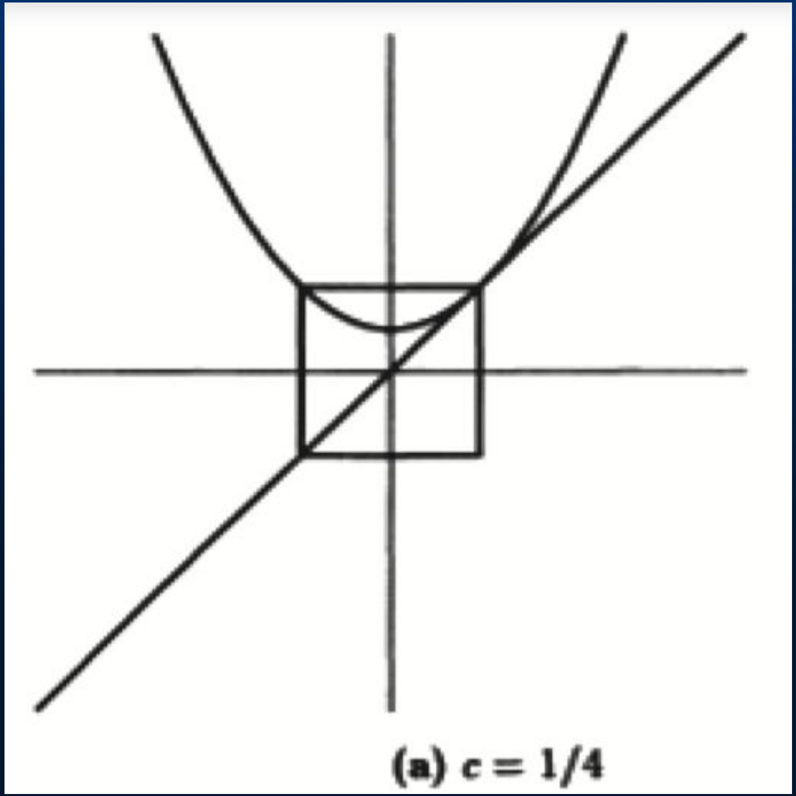


Figure : Graphs of Q_C

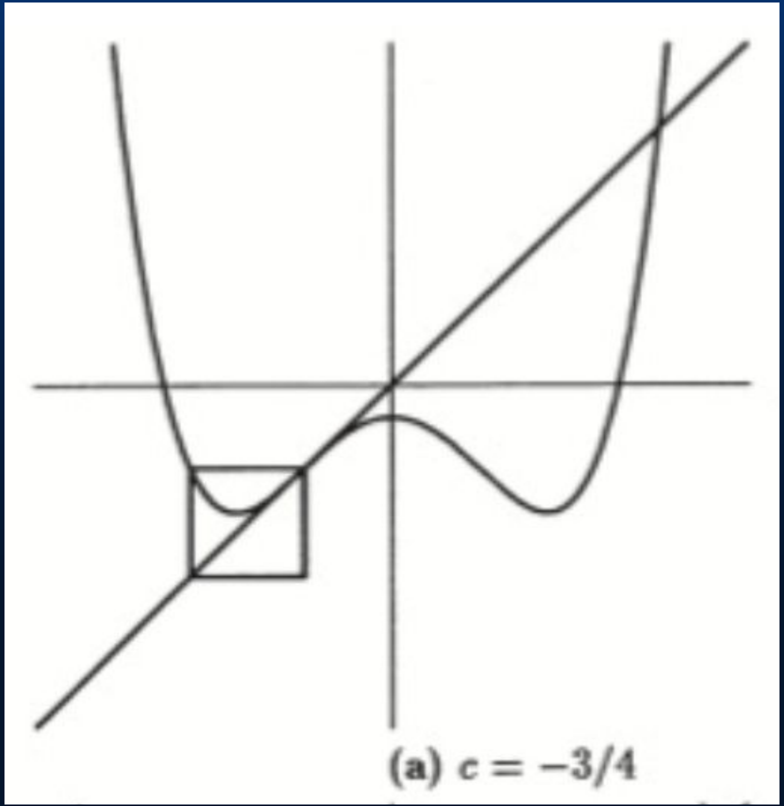


Figure : Graphs of Q^2_C

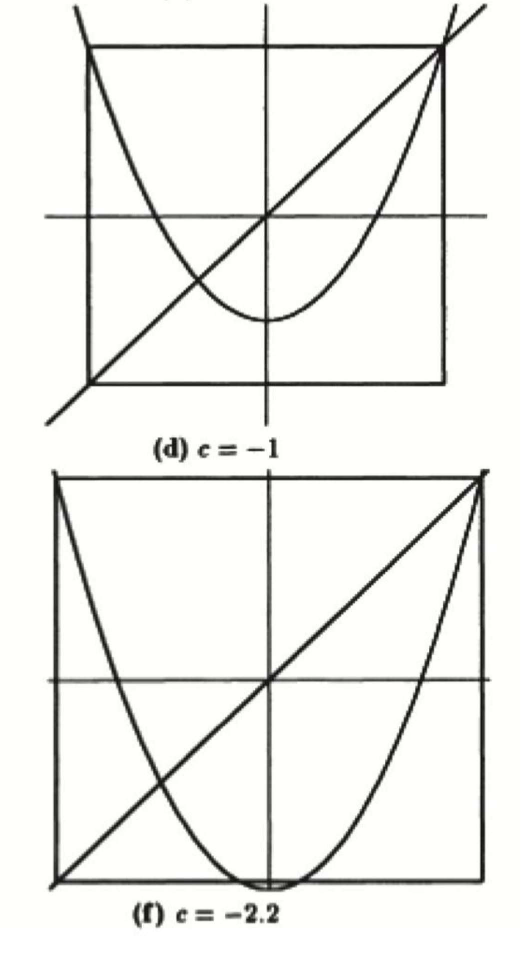


Figure : Graphs of Q_c

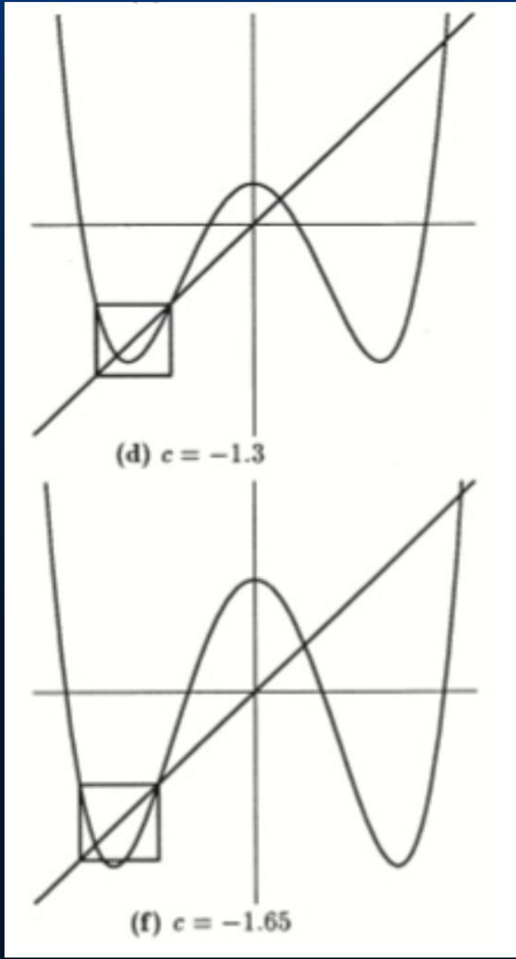


Figure : Graphs of Q^2_c

We can say that the function Q^2_c undergoes a similar sequence of dynamical behaviors on this interval as Qc did on the larger interval (again, because they resemble each other).

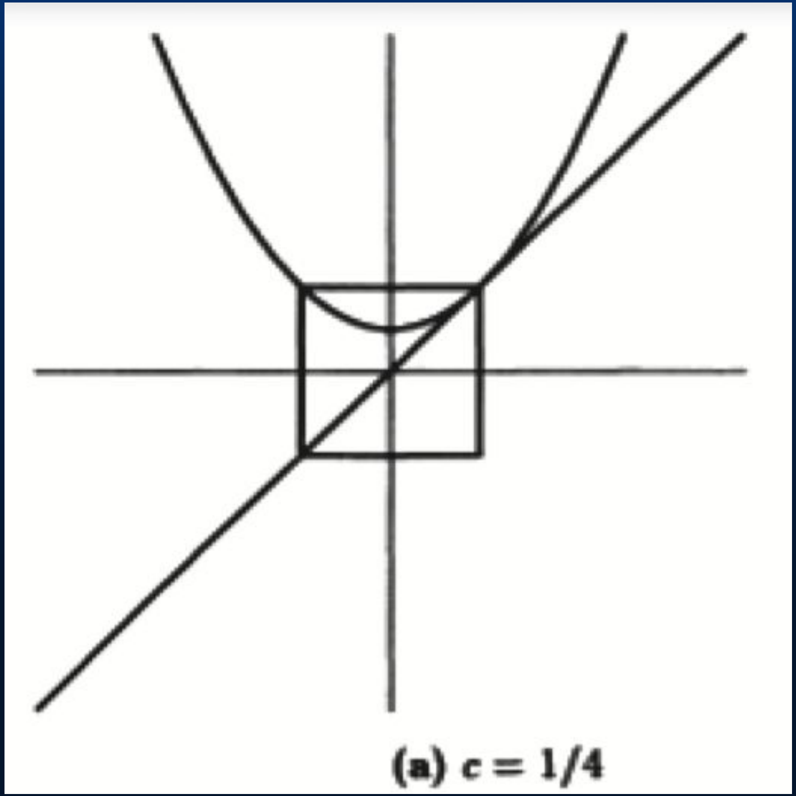
So we can expect a small part of Q^4_c to look similar to Q^2_c

This is the beginning of a process called ***renormalization***

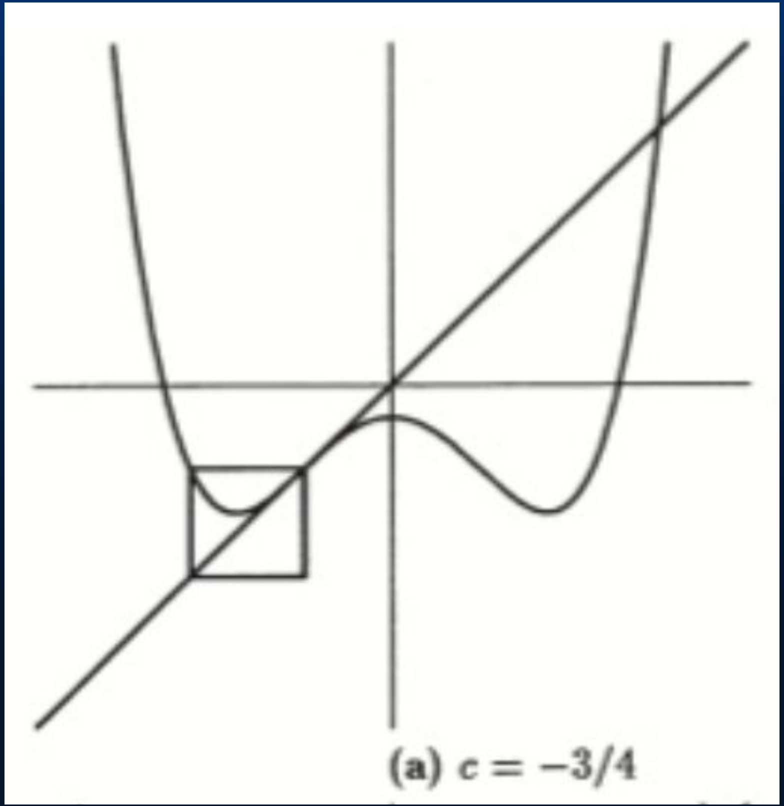
After what we've seen, how do we understand ***renormalization?***

When we zoom into a small subinterval of the graph of the previous stage, the map that we get resembles the previous stage.

At the n^{th} stage, we find a tiny subinterval on which $Q^{2n}{}_c$ resembles the original function. In particular, as c decreases, the graph of $Q^{2n}{}_c$ make the transition from a saddle-node bifurcation , through a period doubling, and on into the ***chaotic*** *regime.*



This right here is a saddle node bifurcation



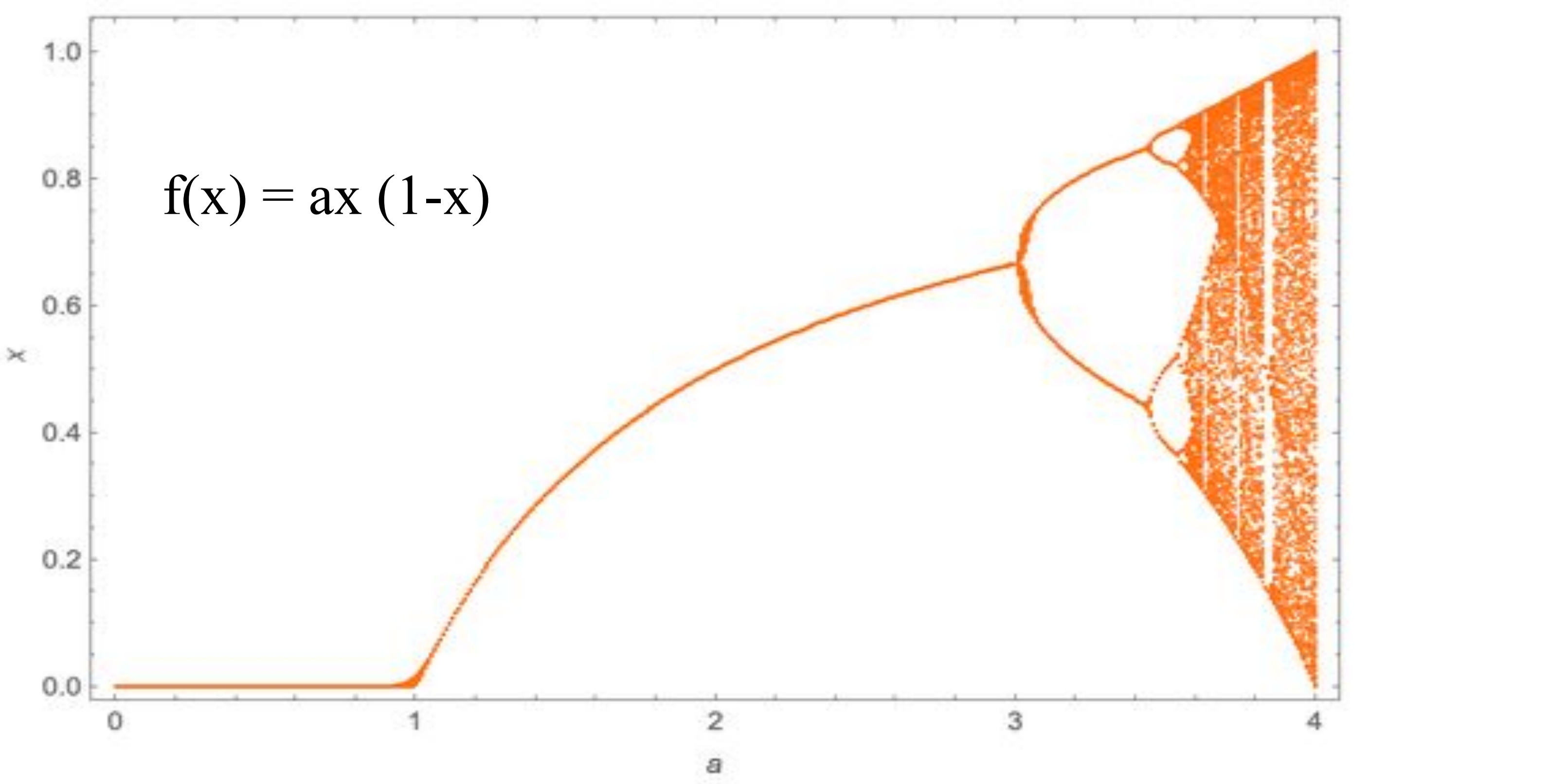
This is a saddle node bifurcation too but in the context of Q^2_c, it is period doubling.

At the n^{th} stage, we find a tiny subinterval on which $Q^{2n}{}_c$ resembles the original function. In particular, as c decreases, ***the graph of $Q^{2n}{}_c$ make the transition from a saddle-node bifurcation , through a period doubling, and on into the chaotic regime.***

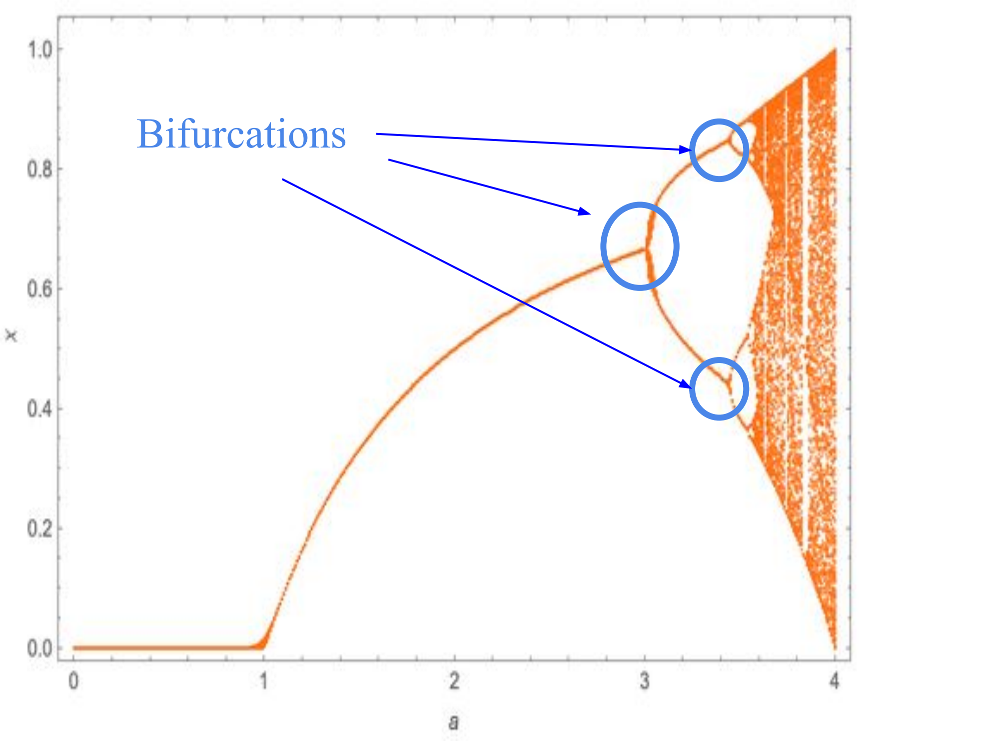
3.

Feigenbaum’s Constant

The Formation of Chaos



Period-Doubling Bifurcation Points



- A period doubling bifurcation occurs when a slight change in a system's parameters causes a new periodic trajectory to emerge from an existing periodic trajectory

- The new one doubles the period of the original .

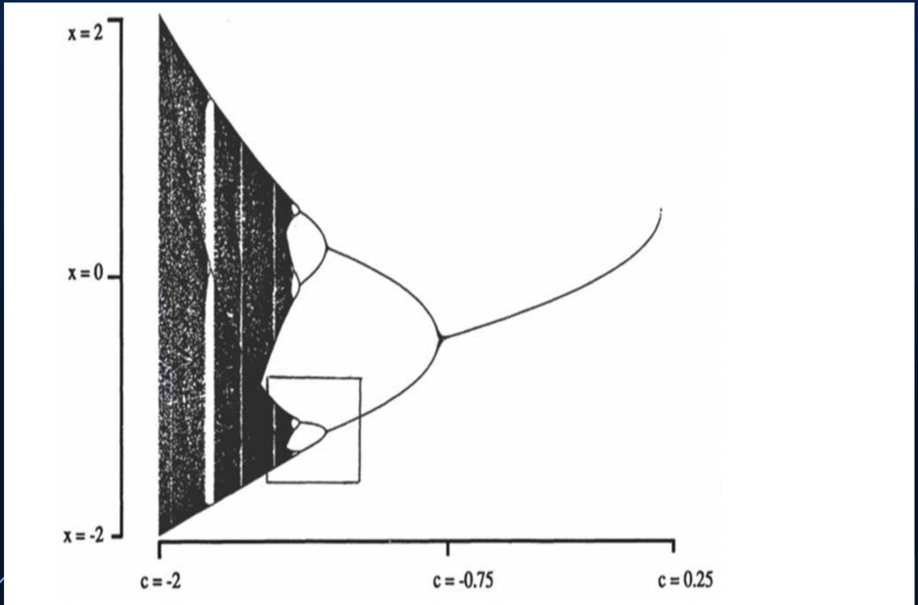
Definition of Feigenbaum Constant

- The Feigenbaum constant is the limiting ratio of each bifurcation interval to the next between every period doubling.
- Given a_n are discrete values of a at the nth period doubling point, the limit is shown as below:

$$\delta = \lim_{n\to\infty} \frac{a_{n-1} - a_{n-2}}{a_n - a_{n-1}} = 4.669\,201\,609\,\ldots,$$

Feigenbaum Constant Exploration

- $F(x) = x^2 + c$



Feigenbaum Constant Exploration

n	Period = 2^n	Bifurcation Value	Ratio = $C_{n-1} - C_{n-2} / C_n - C_{n-2}$
1	2	-0.75	/
2	4	-1.25	/
3	8	-1.3680989	4.2337
4	16	-1.3940462	4.5515
5	32	-1.3996312	4.6639
6	64	-1.4008287	4.6682

$$\delta = \lim_{n\to\infty} \frac{a_{n-1} - a_{n-2}}{a_n - a_{n-1}} = 4.669\,201\,609\ \ldots,$$

Feigenbaum Constant Exploration

STEPS:

1. Compute the first 2^n points on the orbit of the critical point
2. Record the values in tabular form
3. Use calculator to compute the ratios.

4.

Computing Feigenbaum’s Constant

Feigenbaum Constant Exploration

Trying it with hand:

Goal: Finding values of C_i where 0 is a periodic point of $Qc_i\ (x) = x^2 + c$ of prime period 2^i where i = 0,1,2,4,5,6 ...

- > With the c values found, we can compute the ratio between every period doubling

Feigenbaum Constant Exploration

...

-> Because it is a lot of algebra, we can use a MATLAB code to compute the C values for us!

Feigenbaum’s Constant Exploration

```
format long
n = 10; % number of c values to find
c = zeros(1,n);
delta = zeros(1,n-1);
c(1) = 0;
c(2) = -1;
delta(1) = 4;


for j = 2:(n-1)
    alpha_0 = c(j) + (c(j) - c(j-1))/delta(j-1); % initial guess for c
    c(j+1) = approximate(j, alpha_0);
    delta(j) = ((c(j)) - (c(j-1)))/((c(j+1))-(c(j)));
end
delta

function c_value = approximate(i, alpha_0)
    m = 50; % number of steps when approximating a c value
    alpha = alpha_0;
    for j = 1:m
        x = 0; xprime = 0;
        for k = 1:2^i
            xprime = 2*x*xprime + 1;
            x = x^2 + alpha;
        end
        alpha = alpha - x/xprime;
    end
    c_value = alpha;
end
```

Feigenbaum Constant Exploration

... For $x^2 + c$,

```
delta =

  Columns 1 through 6

   4.000000000000000   3.218511422038089   4.385677598568320   4.600949276538136   4.655130495391585   4.666111947823159

  Columns 7 through 9

   4.668548581451485   4.669060660771060   4.669171554366122

fx >> 
```

$$\delta = \lim_{n \to \infty} \frac{a_{n-1} - a_{n-2}}{a_n - a_{n-1}} = 4.669\,201\,609\,\ldots,$$

Feigenbaum Constant Exploration

Conclusion

-> We can see that as we proceed with finding the c-value of the function, the ratio of the intervals between bifurcation points approaches Feigenbaum's constant.

Significance of Feigenbaum's constant

- Universal constant of chaos theory (at first it was only discovered for the logistic maps)
- Feigenbaum's constant appears in problems of fluid-flow turbulence, electronic oscillators, chemical reactions, etc.
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Theorem: If x_0 is an attracting periodic point for F, there is a critical point of F whose orbit is attracted to the orbit of x_0.

This theorem explains why we see at most one attracting periodic orbit for the quadratic family $Q_c(x) = x^2+c$.