Complex Numbers and Julia Sets

MATH 241-Dynamical Systems

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Contents

Introduction

Background

First notion of julia set:

1. 1920's : Gaston Julia and Pierre Fatou
2. But the field died out...
3. Second half of the 20th century : Technology!
4. 1980 : Benoit Mandelbrot plots the Julia and Mandelbrot sets
5. Since then : The field has flourished

Complex numbers and their arithmetic

Complex numbers

Definition : A complex number is a number of the form $x + iy$, with x and y being real numbers and i being the imaginary number.

By definition, $i = \sqrt{-1}$.

The set of all complex numbers can be denoted by the symbol $\mathbb{C}$.

Examples :

1. $\pi - 3i$ is the complex number with real part π and imaginary part -3.
2. $2i$ is the complex number with real part 0 and imaginary part 2. $(0 + 2i)$.
3. 1 is the complex number with real part 1 and imaginary part 0. $(1 + 0i)$

Modulus

Definition : If $z = x + iy$ is a complex number, the modulus $|z|$ is the real number $\sqrt{x^2 + y^2}$.

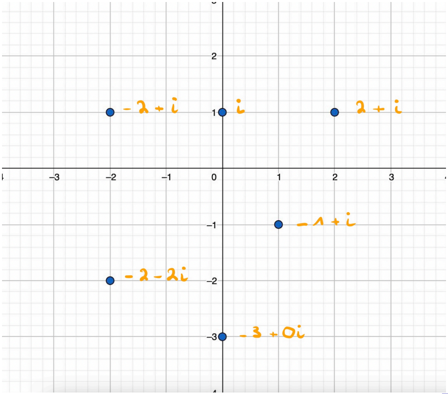
It represents the distance from the origin to z.

Can we visualize that? Yes!

Plotting complex numbers in the plane

To plot a complex number $x + iy$, we simply place it at the point (x, y) in the Cartesian plane.

Examples :



Complex Arithmetic

...So the modulus is computed the exact same way that we would compute the distance from the origin to a point in the Cartesian plane!

Most arithmetic operations are defined for complex numbers in the natural way.

To add : We add two complex numbers by summing their real parts and their imaginary parts. For example,

$$(4+2i)+(3+7i)=7+9i.$$

To multiply : We use the distributive laws recalling that $i^2=-1$. For example,

$$(1+2i)\cdot(2+3i)=1*2+1*3i+2i*2+2i*3i=2+3i+4i+6i^2=2+7i-6=-4+5i$$

Polar representation

An alternative method : Polar representation.

Given a complex number z, its polar representation is determined by the modulus of z, and its polar angle.

Polar angle : The angle between the positive x-axis and the ray from the origin to z measured in the counterclockwise direction.

We can denote the modulus as $r = |z|$ and the polar angle as θ.

The polar representation of $z = x + iy$ is :

$$z = rcos\theta + irsin\theta$$

Polar form

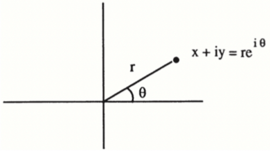
Recall Euler's Formula from calculus :

$$e^{i\theta} = cos\theta + isin\theta$$

So, any complex number may also be written in polar form as :

$$z = re^{i\theta}$$

To sum up,



Note : to multiply two complex numbers, we multiply their moduli and add their polar angles.

Julia Set

Julia Set

According to the book : "The Julia set is the place where all of the chaotic behavior of the complex function occurs". Let us see what that means...

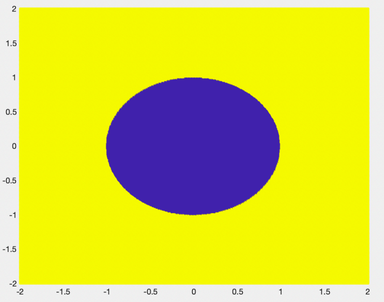
Julia set

We consider quadratic functions of the form

$$Q_c(z) = z^2 + c.$$

Filled Julia Set : For a fixed value c, the filled Julia Set is the set of all seeds whose orbits are bounded (do not go to infinity).

Example, the squaring function : The squaring function has $c = 0$. Its julia set looks like this :



Squaring function

1. In blue : The bounded seeds for which the orbit does not go to infinity (the julia set).
2. In yellow : the seeds for which the orbit "blows up".

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Why does it look like this?

If the initial seed is $z_0 = re^{i\theta}$, after n iterations we get, $z_n = r^{2^n} e^{i(2^n\theta)}$.

No matter what the value of n is, the power of the e function will always be some angle, so we don't need to think about that and can focus on the r^{2^n} part.

Squaring function

1. If r<1 : r^{2^n} goes to 0 as n goes to ∞. So the orbit goes to 0 and is bounded.
2. If r>1 : r^{2^n} goes to ∞ as n goes to ∞. So the orbit goes to ∞ and is unbounded.
3. If r=1 : r^{2^n} doesn't change and stays 1. This case defines the boundary of the filled julia set also known as the julia set.

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Computing the filled julia set

Theorem (The Quadratic Escape Criterion) : The orbit of an initial seed z_0 diverges to ∞ whenever $|z_0|$ exceeds

$$R = max(|c|, 2)$$

(From Basic real and complex dynamics, a computational approach by Mark McClure)

The idea : We consider a rectangular grid of points in some region in the plane. For each point in this grid, we compute the corresponding orbit and ask whether or not this orbit tends to infinity.

Algorithm for the Filled Julia Set : Choose a maximum number of iterations, N. For each point z in the grid, compute the first N points on the orbit of z. If $|Q_c^n(z)| > max(|c|, 2)$ for some $i \leq N$, then stop iterating and color z yellow (the escape criterion is satisfied). If $|Q_c^n(z)| \leq max(|c|, 2)$ for all $i \leq N$, then color z blue (the escape criterion is not satisfied, so the orbit is bounded). Yellow points have orbits that escape, whereas blue points do not, at least for the first N iterations. So the blue points yield an approximation to the filled Julia set.

Observation 1

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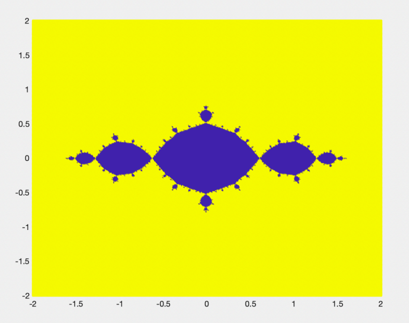
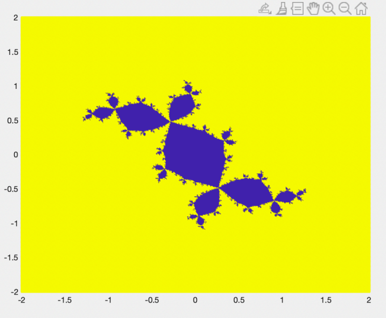
1. For different c-values, K_c can change shapes dramatically. Often, K_c is a connected set in the plane. For $c = -1$:



Observation 2

1. The Julia sets for K_c with c is not 0 or -2 looks like a fractal. For $c = 0.12 - 0.75i$:



Observation 3

1. For some special c-values, K_c is made of disconnected disks and points

After doing some extra research, it turns out that if the julia set looks like a fractal or a connected set, then the seed is in a set called the Mandelbrot set. But that's a topic for another presentation!

Thank you!

Thank you!