Math 301, Spring 2023 — Homework 1

Tim Chumley

Due February 3

Instructions. Please submit your solutions to the following problems on Gradescope. Your proof answers should be written in complete sentences and avoid using symbols like $\Rightarrow$, $\therefore$, or $\because$. Edit rough drafts and reread the guidelines for writing mathematics before submitting. **Please typeset Problems 1 and 2 in LaTeX**. For the rest, you may use LaTeX or submit handwritten solutions. When you submit handwritten solutions, make sure your scan is clear, well-aligned, and as readable as possible. Make sure to select which problem is on each page in Gradescope.

Problem 1. Let $n \geq 1$. Prove by induction that $|x_1+x_2+\cdots+x_n| \leq |x_1|+|x_2|+\cdots+|x_n|$ for any real numbers $x_1, \ldots, x_n$.

Problem 2. Let $S, T \subset \mathbb{R}$ be nonempty, bounded sets with the property $S \subseteq T$.

1. Prove that $\inf T \leq \inf S$.
2. Explain why $\inf T \leq \inf S \leq \sup S \leq \sup T$. Note that the core of this question is to briefly prove the middle inequality.

Problem 3. Let $S, T \subset \mathbb{R}$ be nonempty sets with the property that $s \leq t$ for all $s \in S$ and $t \in T$.

1. Prove that S is bounded above and T is bounded below.
2. Prove that $\sup S \leq \inf T$.
3. Give an example where $\sup S = \inf T$ and $S \cap T = \emptyset$.

Problem 4. Let $A \subset \mathbb{R}$ be a non-empty bounded set and let $c \geq 0$. Define the set cA as follows:

$$cA = \{cx : x \in A\}.$$

Prove that $\sup(cA) = c \sup A$. *Hint: start by proving $c \sup A$ is an upper bound of the set cA and then prove it's the least upper bound of cA.*

Problem 5. For each set below, give its maximum and minimum, if either exists, and its supremum and infimum.

1. $\left\{r \in \mathbb{Q} : r^3 < 8\right\}$
2. $\left\{n + \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$
3. $\bigcap_{n=1}^{\infty}(1 - 1/n, 1 + 1/n)$

Problem 6. For each description below, give an example of a set with the desired property or explain why no such set exists.

1. A set S with $\inf S \geq \sup S$.
2. A set that contains its infimum but not its supremum.
3. A bounded subset of $\mathbb{Q}$ that contains its supremum but not its infimum.