Math 301, Spring 2023 — Homework 10

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Due April 21

Instructions. Please submit your solutions to the following problems on Gradescope. Your proof answers should be written in complete sentences and avoid using symbols like $\Rightarrow$, $\therefore$, or $\because$. Edit rough drafts and reread the guidelines for writing mathematics before submitting. **No LaTeX required this week**. When you submit handwritten solutions, make sure your scan is clear, well-aligned, and as readable as possible. Make sure to select which problem is on each page in Gradescope.

Problem 1. Use the Mean Value Theorem to prove that $|\cos x - \cos y| \leq |x - y|$ for all $x, y \in \mathbb{R}$. You may assume without proof that $\cos x$ is differentiable on $\mathbb{R}$ and its derivative is $-\sin x$.

Problem 2. Let $f : I \to \mathbb{R}$ be differentiable and suppose that $f'(x) \leq 0$ for all $x \in I$. Use the Mean Value Theorem to prove that f is decreasing.

Problem 3. Use the Mean Value Theorem to prove that if f is differentiable on $[a, b]$ and f' is continuous on $[a, b]$ then f is Lipschitz on $[a, b]$.

Problem 4. A function $f : I \to \mathbb{R}$ is called *injective on* I if for every $x, y \in I$ with $x \neq y$, we have $f(x) \neq f(y)$. Use the Mean Value Theorem to prove that if $f'(x) \neq 0$ for all $x \in I$ then it is injective on I.

Problem 5. Let $f : I \to \mathbb{R}$ be differentiable with $f'(x) \neq 1$ for all $x \in I$. Use the Mean Value Theorem to prove that the equation $f(x) = x$ cannot have more than 1 solution in I.

Problem 6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(t) = \begin{cases} t & t < 0 \\ t^2 + 1 & 0 \leq t \leq 2 \\ 0 & t > 2. \end{cases}$$

a. Find a formula for $F(x) = \int_0^x f(t)\, dt$ for all $x \in \mathbb{R}$

b. Give ϵ-δ proofs to show that F is continuous at $x = 0$ and $x = 2$.

Problem 7. Compute the following limits.

a. $\lim_{x \to 0} \frac{1}{x} \int_0^x e^{t^2}\, dt$

b. $\lim_{h \to 0} \frac{1}{h} \int_3^{3+h} e^{t^2}\, dt$