Math 301, Spring 2023 — Homework 11

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Due April 28

Instructions. Please submit your solutions to the following problems on Gradescope. Your proof answers should be written in complete sentences and avoid using symbols like $\Rightarrow$, $\therefore$, or $\because$. Edit rough drafts and reread the guidelines for writing mathematics before submitting. **No LaTeX required this week**. When you submit handwritten solutions, make sure your scan is clear, well-aligned, and as readable as possible. Make sure to select which problem is on each page in Gradescope.

Problem 1. Suppose that (f_n) is a sequence of functions, each of which is bounded on D. Prove that if $f_n \to f$ uniformly on D then f is bounded on D.

Problem 2. Suppose that (f_n) is a sequence of functions, each of which is uniformly continuous on (a,b). Prove that if $f_n \to f$ uniformly on (a,b) then f is uniformly continuous on (a,b).

Problem 3. Let $f_n : \mathbb{R} \to \mathbb{R}$ be given by $f_n(x) = \frac{1+\sin(nx)}{n^2}$ for each $n \geq 1$. Find $f(x) = \lim_{n\to\infty} f_n(x)$ and prove that $f_n \to f$ uniformly on $\mathbb{R}$.

Problem 4. Let $f_n : [0,\infty) \to \mathbb{R}$ be given by $f_n(x) = \frac{x^n}{n+x^n}$.

a. Find $f(x) = \lim_{n\to\infty} f_n(x)$. *Hint: you should get a piecewise defined function.*

b. Determine whether $f_n \to f$ uniformly on $[0,1]$ and justify your assertion. *Hint: begin by explaining why $|f_n(x)| \leq 1/n$ for all $x \in [0,1]$.*

c. Determine whether $f_n \to f$ uniformly on $[0,\infty)$ and justify your assertion. *Hint: a very short proof is possible.*

Problem 5. Let $f_n : [0,1]$ be given by

$$f_n(x) = \begin{cases} n & x \in \left(0, \frac{1}{n}\right) \\ 0 & \text{otherwise} \end{cases}$$

for all $n \geq 1$.

a. Find $f(x) = \lim_{n\to\infty} f_n(x)$.

b. Let $n \geq 1$. Compute $\int_0^1 f_n(x)\,dx$ in terms of n.

c. Determine whether $f_n \to f$ uniformly on $[0,1]$ and justify your assertion.

Problem 6. Let $f_n : \mathbb{R} \to \mathbb{R}$ be given by $f_n(x) = \frac{n+\cos x}{2n+\sin^2 x}$.

a. Find the pointwise limit $f(x) = \lim f_n(x)$.

b. Prove that $f_n \to f$ uniformly on $\mathbb{R}$.

c. Compute $\lim_{n\to\infty} \int_2^7 f_n(x)\,dx$ without integrating f_n.

Problem 7. Let $f_n : \mathbb{R} \to \mathbb{R}$ be given by $f_n(x) = \frac{nx}{1+nx^2}$.

a. Find the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$.

b. Determine whether $f_n \to f$ uniformly on $[0,1]$ by computing $\lim_{n\to\infty} \sup \{|f_n(x) - f(x)| : x \in [0,1]\}$.

c. Determine whether $f_n \to f$ uniformly on $[1,\infty)$ by computing $\lim_{n\to\infty} \sup \{|f_n(x) - f(x)| : x \in [1,\infty)\}$.