Math 301, Fall 2021 — Homework 2

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Due September 17

Instructions. Please submit your solutions to the following problems on Gradescope. For the group part, please type your answers in LaTeX and submit the output PDF. You'll only submit one write-up for the whole group. For the solo part, you may handwrite solutions or use LaTeX. Make sure to select which problem is on each page in Gradescope.

Groups

Groups will be posted on the class Moodle page.

Some comments and advice on group work

- I've tried to use your given preferences in forming groups, but please let me know if I missed anything or if I can help talk through any issues.
- I'll try to hold some class time for you to work together and I've asked for one of the evening help rooms to be held as a meeting spot for our class. I'm waiting to hear back on a day and time.
- You should try to meet at least once as a whole group to make sure everyone's input has been heard on the final write up. Feel free to meet more than that, either as a whole group, or in subsets as you work on problems. Math is easier and fun when you struggle through it together, and I think sometimes it takes some time to let your guard down when working with others.

Group problem

Problem 1. Let (s_n) be a sequence such that $\lim s_n = 0$. Let (t_n) be a bounded sequence (this means there exists a constant $M > 0$ such that $|t_n| \leq M$ for all $n \in \mathbb{N}$). Prove that $\lim s_n t_n = 0$.

Solo problems

Problem 2. Let A and B be a nonempty bounded subsets of $\mathbb{R}$. Define a new set by

$$A + B = \{a + b : a \in A, b \in B\}.$$

Try each of the following problems, making sure to do them in order.

1. Prove that for any $b \in B$, $\sup(A + B) - b$ is an upper bound for A.
2. Prove that $\sup A \leq \sup(A + B) - b$.

3. Prove that $\sup B \leq \sup(A+B) - \sup A$.

4. Prove that $\sup(A+B) \leq \sup A + \sup B$.

5. Explain why $\sup(A+B) = \sup A + \sup B$.

Problem 3. Let $a \in \mathbb{R}$ and consider the set $A = \{r \in \mathbb{Q} : r < a\}$. We've thought about sets like this in homework and worksheets before and used our intuition to say that $\sup A = a$. We can *prove* this is true now using the density of $\mathbb{Q}$ in $\mathbb{R}$.

1. Explain why $\sup A \leq a$.

2. Prove that $\sup A = a$ using a proof by contradiction.

Problem 4. For each of the following sequences, determine its limit, or state that the limit does not exist and then give a proof of your claim using an ϵ-N proof if it converges and a contradiction proof if it diverges (like in Section 8).

1. $a_n = \frac{7n-19}{3n+7}$

2. $b_n = \frac{n+6}{n^2+3}$

3. $c_n = \sin(2n\pi/3)$

Problem 5. Let (s_n) be a sequence that converges. Show that if $s_n \geq a$ for all but finitely many n, then $\lim s_n \geq a$.

Problem 6. Let's consider two new definitions.

1. A sequence (s_n) is *eventually* in a set $A \subseteq \mathbb{R}$ if there exists $N > 0$ such that $a_n \in A$ for all $n \geq N$.

2. A sequence (s_n) is *frequently* in a set $A \subseteq \mathbb{R}$ if for every $N > 0$ there exists $n \geq N$ such that $a_n \in A$.

   (a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?

   (b) Which definition is stronger? Does frequently imply eventually or does eventually imply frequently?

   (c) Give a rephrasing of the definition of *convergence* of a sequence (s_n) to L using one of these new terms. Should we use eventually or frequently?

   (d) Suppose (s_n) is a sequence with infinitely many terms equal to 2. Does that necessarily mean (s_n) is eventually in the interval $(1.99, 2.01)$? Is it necessarily frequently in $(1.99, 2.01)$?