Math 301, Spring 2023 — Homework 2

Tim Chumley

Due February 10

Instructions. Please submit your solutions to the following problems on Gradescope. Your proof answers should be written in complete sentences and avoid using symbols like $\Rightarrow$, $\therefore$, or $\because$. Edit rough drafts and reread the guidelines for writing mathematics before submitting. **Please typeset Problem 3 in LaTeX**. For the rest, you may use LaTeX or submit handwritten solutions. When you submit handwritten solutions, make sure your scan is clear, well-aligned, and as readable as possible. Make sure to select which problem is on each page in Gradescope.

Problem 1. Prove that if $a > 0$, then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < a < n$.

Remark. You'll use the two corollaries to the Archimedean property to do this problem, but be careful to note that there is a subtlety that means you have to do one extra step beyond citing the corollaries.

Problem 2. Let $a \in \mathbb{R}$ and consider the set $A = \{r \in \mathbb{Q} : r < a\}$. Our intuition tells us that $\sup A = a$, but we can *prove* this is true now using the density of $\mathbb{Q}$ in $\mathbb{R}$.

a. Explain why $\sup A \leq a$ using the definition of supremum.

b. Prove that $a \leq \sup A$ using a proof by contradiction and the density of $\mathbb{Q}$ in $\mathbb{R}$.

Problem 3. For each of the following sequences, determine its limit, or state that the limit does not exist and then give a proof of your claim using an ϵ-N proof if it converges and a contradiction proof if it diverges (like in Section 8).

a. $a_n = \frac{7n-19}{3n+7}$

b. $b_n = \frac{n+6}{n^3-4}$

c. $c_n = \frac{4n^2+3}{3n^2-2}$

d. $d_n = \sin(2n\pi/3)$

Problem 4. Let's consider two new definitions.

- A sequence (a_n) is *eventually* in a set $A \subseteq \mathbb{R}$ if there exists $N > 0$ such that $a_n \in A$ for all $n > N$.
- A sequence (a_n) is *frequently* in a set $A \subseteq \mathbb{R}$ if for every $N > 0$ such that $a_n \in A$ for some $n > N$.

a. Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?

b. Give an example of a sequence (a_n) that has infinitely many terms equal to 2 and is eventually in the interval $(1.99, 2.01)$.

c. Give an example of a sequence (a_n) that has infinitely many terms equal to 2 and is frequently, but not eventually, in the interval $(1.99, 2.01)$.

d. Does frequently imply eventually or does eventually imply frequently?

e. Give a rephrasing of the definition of (a_n) *converges to* L using one of these new definitions. Should we use eventually or frequently? Your rephrasing should still start with *for all* $\epsilon > 0$....

Problem 5. Let $S \subset \mathbb{R}$ be a nonempty set that is bounded above and let $\alpha = \sup S$. We've proven (on day 3 of lecture) that for every $\epsilon > 0$ there exists $x \in S$ such that $x > \alpha - \epsilon$. You'll use this fact in the following problem; please cite it as the Supremum Lemma in this problem when you use it.

a. Explain why for each $\epsilon > 0$ there exists a corresponding $a \in S$ such that $\alpha - \epsilon \leq a \leq \alpha$.

b. Explain why there exists an element $a_1 \in S$ such that $\alpha - 1 \leq a_1 \leq \alpha$.

c. Give an explanation using the Supremum Lemma for why there exists (ie. why it's theoretically possible to construct) an infinite list (ie. a sequence) of elements $a_1, a_2, \ldots$ from S such that $\alpha - 1/n \leq a_n \leq \alpha$ for each $n \geq 1$.

d. What does the sequence constructed in part c. converge to? Prove your claim.

e. Why did we need to use the Supremum Lemma to construct the sequence in part c. instead of just using $a_n = \alpha - 1/n$?