Math 301, Fall 2021 — Homework 3

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Instructions. Please submit your solutions to the following problems on Gradescope. For the group part, please type your answers in LaTeX and submit the output PDF. You'll only submit one write-up for the whole group. For the solo part, you may handwrite solutions or use LaTeX. Make sure to select which problem is on each page in Gradescope.

Groups

We'll use the same groups as last time.

Group problem

Problem 1. For each of the following, give an example or explain why it's impossible to give an example.

1. Sequences (a_n) and (b_n) which both diverge but whose sum $(a_n + b_n)$ converges.
2. Sequences (a_n) and (b_n) where (a_n) converges, (b_n) diverges, and (a_n+b_n) converges.
3. A convergent sequence (a_n) where $a_n \neq 0$ for all n and $(1/a_n)$ diverges.
4. An unbounded sequence (a_n) and a convergent sequence (b_n) where $(a_n - b_n)$ is bounded.
5. Sequences (a_n) and (b_n) where $(a_n b_n)$ and (a_n) converge, but (b_n) diverges.

Solo problems

Problem 2. Suppose that $\lim a_n = 0$ and $|b_n - b| \leq a_n$ for all $n \geq 1$. Prove that $\lim b_n = b$.

Problem 3. Let $s_1 = 1$ and for $n \geq 1$, let $s_{n+1} = \sqrt{s_n + 1}$.

1. Write out the first 4 terms of (s_n).
2. It turns out that the sequence (s_n) converges. Assume this fact and use the algebraic limit theorems to prove that the limit is $\frac{1}{2}(1 + \sqrt{5})$.

Problem 4. When we prove a sequence (a_n) is unbounded and diverges to $+\infty$, written as $\lim_{n\to\infty} a_n = +\infty$, we must show that for every $M > 0$, there exists N so that $n > N$ implies $a_n > M$. For example, to prove that $\lim_{n\to\infty} n^2 = +\infty$ we could write:

Proof. Let $M > 0$. Our aim is to show that there exists $N \in \mathbb{N}$ such that $n^2 > M$ for all $n > N$. Observe that if we let $N = \sqrt{M}$, then for all $n > N$ we have

$$n^2 > N^2 = (\sqrt{M})^2 = M.$$

$\square$

The idea in finding N in the proof above is to solve for N in the inequality $N^2 > M$, much like we solve for N in ϵ-N convergence proofs. Give a similar proof to show that $\lim_{n\to\infty}(1.01)^n = +\infty$.

Problem 5. Show that if $\lim s_n = +\infty$ and $k > 0$, then $\lim ks_n = +\infty$.

Problem 6. Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$.

1. Find s_2, s_3, and s_4.
2. Use induction to show that $s_n > 1/2$ for all $n \geq 1$.
3. Show that (s_n) is a decreasing sequence.
4. Explain why (s_n) converges and find $\lim_{n\to\infty} s_n$.

Problem 7. Prove that if (a_n) is a bounded, decreasing sequence, then a_n converges. That is, prove the second case of the monotone convergence theorem.