Math 301, Spring 2023 — Homework 3

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Due February 17

Instructions. Please submit your solutions to the following problems on Gradescope. Your proof answers should be written in complete sentences and avoid using symbols like $\Rightarrow$, $\therefore$, or $\because$. Edit rough drafts and reread the guidelines for writing mathematics before submitting. **Please typeset Problem 7 in LaTeX**. For the rest, you may use LaTeX or submit handwritten solutions. When you submit handwritten solutions, make sure your scan is clear, well-aligned, and as readable as possible. Make sure to select which problem is on each page in Gradescope.

Problem 1. Suppose that $\lim_{n\to\infty} a_n = 0$ and $|b_n - b| \leq a_n$ for all $n \geq 1$. Prove that $\lim_{n\to\infty} b_n = b$.

Problem 2. Let (a_n) be a sequence that converges to 0 and let (b_n) be a bounded sequence. Prove that $(a_n b_n)$ converges to 0.

Problem 3. A sequence (a_n) is said to be *unbounded and diverge to* $+\infty$, written as $\lim_{n\to\infty} a_n = +\infty$, if for every $M > 0$, there exists $N > 0$ so that $a_n > M$ when $n > N$. For example, to prove that $\lim_{n\to\infty} n^2 = +\infty$ we could write:

Proof. Let $M > 0$. Our aim is to show that there exists $N \in \mathbb{N}$ such that $n^2 > M$ for all $n > N$. Observe that if we let $N = \sqrt{M}$, then for all $n > N$ we have

$$n^2 > N^2 = (\sqrt{M})^2 = M.$$

$\square$

The idea in finding N in the proof above is to solve for N in the inequality $N^2 > M$, much like we solve for N in ϵ-N convergence proofs. Give a similar proof to show that $\lim_{n\to\infty}(1.01)^n = +\infty$.

Problem 4. Suppose $\lim_{n\to\infty} a_n = +\infty$ and $k > 0$. Prove that $\lim ka_n = +\infty$.

Problem 5. For each of the following, give an example or explain why it's impossible to give an example.

a. Sequences (a_n) and (b_n) which both diverge but whose sum $(a_n + b_n)$ converges.

b. Sequences (a_n) and (b_n) where (a_n) converges, (b_n) diverges, and (a_n+b_n) converges.

c. A convergent sequence (a_n) where $a_n \neq 0$ for all n and $(1/a_n)$ diverges.

d. Sequences (a_n) and (b_n) where $(a_n b_n)$ and (a_n) converge, but (b_n) diverges.

Problem 6. Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$.

a. Find s_2, s_3, and s_4.

b. Use induction to show that $s_n > 1/2$ for all $n \geq 1$.

c. Show that (s_n) is a decreasing sequence.

d. Explain why (s_n) converges and find $\lim_{n\to\infty} s_n$.

Problem 7. Prove that if (a_n) is a bounded, decreasing sequence, then a_n converges. That is, prove the second case of the monotone convergence theorem.