Math 301, Spring 2023 — Homework 4

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Due February 24

Instructions. Please submit your solutions to the following problems on Gradescope. Your proof answers should be written in complete sentences and avoid using symbols like $\Rightarrow$, $\therefore$, or $\because$. Edit rough drafts and reread the guidelines for writing mathematics before submitting. **Please typeset Problems 4 and 5 in LaTeX**. For the rest, you may use LaTeX or submit handwritten solutions. When you submit handwritten solutions, make sure your scan is clear, well-aligned, and as readable as possible. Make sure to select which problem is on each page in Gradescope.

Problem 1. Let (a_n) be a sequence such that $\lim_{n\to\infty} a_n = +\infty$. Prove that for every subsequence (a_{n_k}) of (a_n), $\lim_{k\to\infty} a_{n_k} = +\infty$.

Problem 2. Let $[a, b]$ be a given closed interval. Suppose that (a_n) is a sequence such that $a_n \in [a, b]$ for all $n \geq 1$. The Bolzano-Weierstrass Theorem and Order Limit Theorems can be used to prove that there exists a convergent subsequence of (a_n) that converges to a value in $[a, b]$. Does this result still hold if we replace $[a, b]$ with the open interval (a, b)? If you believe it does, explain why. If you believe it does not, give an example of a sequence (a_n) and an open interval (a, b) that serve as a counterexample.

Problem 3. For each of the following statements, decide whether it is true or false and give a brief justification.

a. There exists a sequence that does not contain 0 or 1 as terms but contains subsequences that converge to each of these values.

b. If (a_n) has a divergent subsequence, then (a_n) diverges.

c. There exists a sequence (a_n) that has a bounded subsequence but no subsequence that converges.

Problem 4. Suppose that (a_n) and (b_n) are Cauchy sequences. Prove that $(a_n + b_n)$ is a Cauchy sequence using the definition of a Cauchy sequence. Do not use the theorem that says a sequence is Cauchy if and only if it converges.

Problem 5. Suppose that (a_n) and (b_n) are Cauchy sequences. Prove that $(a_n b_n)$ is a Cauchy sequence using the definition of a Cauchy sequence. Do not use the theorem that says a sequence is Cauchy if and only if it converges.

Problem 6. For each of the following sequences, find the limit inferior and limit superior.

a. $a_n = \sin(2n\pi/3)$

b. $w_n = (-2)^n$

c. $x_n = 5^{(-1)^n}$

d. $y_n = (-1)^n + 1/n$

e. $z_n = n \cos\left(\frac{n\pi}{4}\right)$

Problem 7. Let A be a non-empty bounded set and define the set $-A = \{-x : x \in A\}$. Prove that $\sup(-A) = -\inf A$. *Hint: prove that* $-\inf A$ *is an upper bound of* $-A$ *and then prove that* $-\inf A \leq u$ *for any upper bound* u *of* $-A$.

Problem 8. Let (a_n) be a bounded sequence. Prove that $\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty} a_n$.