Math 301, Fall 2021 — Homework 4

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Due October 1

Instructions. Please submit your solutions to the following problems on Gradescope. For the group part, please type your answers in LaTeX and submit the output PDF. You'll only submit one write-up for the whole group. For the solo part, you may handwrite solutions or use LaTeX. Make sure to select which problem is on each page in Gradescope.

Groups

We'll use the same groups as last time.

Group problem

Problem 1. For each of the following statements, decide whether it is true or false, giving a brief justification.

1. There exists a sequence that does not contain 0 or 1 as terms but contains subsequences that converge to each of these values.
2. If (a_n) has a divergent subsequence, then (a_n) diverges.
3. There exists a sequence (a_n) that has a bounded subsequence but no subsequence that converges.
4. If every subsequence of (a_n) converges, then (a_n) converges.
5. Every monotone sequence contains a convergent subsequence.

Solo problems

Problem 2. Consider the following sequences:

$$w_n = (-2)^n, \quad x_n = 5^{(-1)^n}, \quad y_n = 1 + (-1)^n, \quad z_n = n \cos\left(\frac{n\pi}{4}\right).$$

1. For each sequence, give an example of a monotone subsequence.
2. For each sequence, give the possible values that any convergent subsequence can converge to (called the subsequential limits).
3. Which sequences are bounded?
4. Which sequences converge? Diverge to $+\infty$? Diverge to $-\infty$?

Problem 3. Let $[a, b]$ be a given closed interval. Suppose that (a_n) is a sequence such that $a_n \in [a, b]$ for all $n \geq 1$.

1. Use the Bolzano-Weierstrass Theorem and Order Limit Theorems to prove that there exists a convergent subsequence of (a_n) that converges to a value in $[a, b]$.

2. Does the result still hold if we replace $[a, b]$ with the open interval (a, b)? If you believe it does, explain why. If you believe it does not, give an example a sequence (a_n) and an open interval (a, b) that serves as a counterexample.

Problem 4. Let (a_n) be a Cauchy sequence and suppose that $a_n = a$ for infinitely many n. Prove that (a_n) converges to a.

Problem 5. Suppose that (a_n) and (b_n) are Cauchy sequences. Prove that $(a_n + b_n)$ is a Cauchy sequence using only the definition of a Cauchy sequence, but do not use the Algebraic Limit Theorems.

Problem 6. Suppose that (a_n) and (b_n) are Cauchy sequences. Prove that $(a_n b_n)$ is a Cauchy sequence using only the definition of a Cauchy sequence and the fact that Cauchy sequences are bounded, but do not use the Algebraic Limit Theorems.

Problem 7. Please give me some feedback about the group homework. Are you happy with your group or would you like to have the groups change? Do you find the activity beneficial to learning class material? Do you find the class time for group work productive? Do you have any suggestions or other feedback?