Math 301, Spring 2025 — Homework 4

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Due February 28 at 5:00 pm

Instructions. This problem set contains problems mostly from Week 4 of class. The problem numbers refer to our textbook, *Understanding Analysis* by Stephen Abbott.

Problem 1. Please do the following textbook problems: Exercises 2.4.2, 2.4.8bc, 2.5.1ab, 2.5.5.

Remark 1. In Exercise 2.4.2b, please prove the sequence converges if you believe the strategy works and compute the limit.

Remark 2. Here's a hint to thinking about 2.5.5: assume by way of contradiction that (a_n) does not converge to a. This means there exists $\epsilon_0 > 0$ such that for each $N \in \mathbb{N}$ there exists $n \geq N$ such that $|a_n - a| \geq \epsilon_0$. Use this to build a subsequence of (a_n).

Problem 2. Prove that if (a_n) is a bounded, decreasing sequence, then (a_n) converges. That is, prove the second case of the monotone convergence theorem.

Problem 3. Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for each $n \in \mathbb{N}$.

a. Prove that $s_n > 1/2$ for all $n \in \mathbb{N}$.

b. Prove that (s_n) is decreasing.

c. Prove that (s_n) converges and compute $\lim_{n\to\infty} s_n$ and $\inf\{s_n : n \in \mathbb{N}\}$.

Problem 4. Let (a_n) be a sequence such that diverges to $+\infty$. Prove that every subsequence of (a_n) must diverge to $+\infty$ as well.

Remark 3. Recall that a sequence (a_n) diverges to $+\infty$ if for every $M > 0$ there exists $N \in \mathbb{N}$ such that $a_n > M$ for all $n \geq N$.

Problem 5. Let $[a, b]$ be a given closed interval. Suppose that (a_n) is a sequence such that $a_n \in [a, b]$ for all $n \in \mathbb{N}$.

a. Prove that (a_n) has a convergent subsequence whose limit is an element of $[a, b]$.

b. Does this result still hold if we replace $[a, b]$ with the open interval (a, b)? If you believe it does, explain why. If you believe it does not, give a sequence (a_n) and an open interval (a, b) that serve as a counterexample.