Math 301, Spring 2023 — Homework 5

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Due March 3

Instructions. Please submit your solutions to the following problems on Gradescope. Your proof answers should be written in complete sentences and avoid using symbols like $\Rightarrow$, $\therefore$, or $\because$. Edit rough drafts and reread the guidelines for writing mathematics before submitting. **Please typeset Problems 6 and 7 in LaTeX**. For the rest, you may use LaTeX or submit handwritten solutions. When you submit handwritten solutions, make sure your scan is clear, well-aligned, and as readable as possible. Make sure to select which problem is on each page in Gradescope.

Problem 1. Let (s_n) be the sequence of numbers in Figure 11.2 of our textbook, listed in the indicated order.

a. Find the set S of subsequential limits of (s_n).

b. Find $\limsup s_n$ and $\liminf s_n$.

Problem 2. Let (s_n) and (t_n) be bounded sequences.

a. Prove that for any $N \geq 1$

$$\sup\{s_n + t_n : n > N\} \leq \sup\{s_n : n > N\} + \sup\{t_n : n > N\}.$$

Hint: try to start by showing that for any $N \geq 1$, $\sup\{s_n : n > N\} + \sup\{t_n : n > N\}$ is an upper bound for the set $\{s_n + t_n : n > N\}$.

b. Prove that $\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n$.

Problem 3. Prove that if $\limsup |a_n| < \infty$ then (a_n) is bounded. *Hint: let $M_1 = \limsup |a_n|$ and try to bound (a_n) using a constant defined in terms of M_1.*

Problem 4. Suppose $\sum a_n = A$ and $\sum b_n = B$. Use the algebraic limit theorems to prove that

a. $\sum(a_n + b_n) = A + B$,

b. $\sum k a_n = kA$ for any $k \in \mathbb{R}$.

Problem 5. For each of the following series, determine whether it converges and justify your answer.

a. $\sum_{n=1}^{\infty} \frac{n+3}{4n+7}$

b. $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \left(1 + \frac{5}{n}\right)$ *Hint: use the comparison test and Problem 4.*

c. $\sum_{n=1}^{\infty} \frac{n-1}{n^2}$ *Hint: use the fact that $n^2 \geq 2n$ for all $n \geq 2$ and the comparison test.*

Problem 6. Use the Cauchy criterion to prove that if $\sum |a_n|$ converges and (b_n) is bounded, then $\sum a_n b_n$ converges.

Problem 7. Prove that if $a_n \geq 0$ and $\sum a_n$ converges, then $\sum a_n^2$ converges. *Hint: use Problem 6.*

Problem 8. Let (a_n) be a sequence such that

$$|a_{n+1} - a_n| < 2^{-n}$$

for all $n \geq 1$. Prove that (a_n) is a Cauchy sequence. *Hint: remember that* $\sum 2^{-n}$ *converges since it's a geometric series, and so it satisfies the Cauchy criterion for series.*