Math 301, Spring 2023 — Homework 7

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Due March 31

Instructions. Please submit your solutions to the following problems on Gradescope. Your proof answers should be written in complete sentences and avoid using symbols like $\Rightarrow$, $\therefore$, or $\because$. Edit rough drafts and reread the guidelines for writing mathematics before submitting. **Please typeset Problem 1 with LaTeX**. When you submit handwritten solutions, make sure your scan is clear, well-aligned, and as readable as possible. Make sure to select which problem is on each page in Gradescope.

Problem 1. Give an ϵ-δ proof to show that each of the following functions is continuous at the given value a.

a. $f(x) = x^2, x_0 = 5$

b. $f(x) = x^4, a = 1$

c. $f(x) = 1/x, a = 1$

Problem 2. Give an ϵ-δ proof showing that

$$f(x) = \begin{cases} \sqrt{x}\sin\left(\frac{1}{x}\right) & x > 0 \\ 0 & x \leq 0 \end{cases}$$

is continuous at 0. *Hint: your proof should be split into cases involving $x > 0$ and $x < 0$.*

Problem 3. Prove that the following functions are *not* continuous at 0.

a. $f(x) = \begin{cases} x^2 & x \neq 0 \\ 1 & x = 0 \end{cases};$

b. $g(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

c. $h(x) = \begin{cases} 1/x & x \neq 0 \\ 0 & x = 0. \end{cases}$

Problem 4. Prove that the function $f(x) = x^4$ is continuous using the ϵ-δ definition of continuity.

Problem 5. Prove that the function $f(x) = x^4$ is continuous using the sequential condition for continuity (the theorem we proved on Wednesday of Week 9).

Problem 6. Take as given that $f(x) = x^n$ is continuous for any integer $n \geq 0$. Prove that any polynomial is continuous using the sequential condition for continuity and the Algebraic Limit Theorem for sequences. That is, prove that for any integer $m \geq 0$ and constants $c_0, \ldots, c_m \in \mathbb{R}$, the function $p(x) = c_m x^m + c_{m-1}x^{m-1} + \cdots + c_1 x + c_0$ is continuous.

Problem 7. Take as given the result of Problem 5. Prove that any rational function is continuous at all points where its denominator is non-zero using the sequential condition for continuity and the Algebraic Limit Theorem for sequences. That is, prove that for any polynomials $p(x)$ and $q(x)$, the function $r(x) = p(x)/q(x)$ is continuous at any $a \in \mathbb{R}$ where $q(a) \neq 0$.