Math 301, Fall 2021 — Homework 7

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Due October 29

Instructions. Please submit your solutions to the following problems on Gradescope. For the group part, please type your answers in LaTeX and submit the output PDF. You'll only submit one write-up for the whole group. **A small change in group work:** for redos on the group problem, I'll ask you to submit redos individually rather than as a group. For the other problems, you may handwrite solutions or use LaTeX. Make sure to select which problem is on each page in Gradescope.

Group problem

Problem 1. Give an ϵ-δ proof to show that each of the following functions is continuous at the given value x_0.

1. $f(x) = x^2, x_0 = 5$
2. $f(x) = x^3, x_0 = 1$ *Hint: remember that* $x^3 - 1 = (x-1)(x^2+x+1)$.
3. $f(x) = 1/x, x_0 = 2$
4. $f(x) = \begin{cases} \sqrt{x}\sin\left(\frac{1}{x}\right) & x > 0 \\ 0 & x \leq 0 \end{cases}; x_0 = 0$ *Hint: in your proof, you should consider the cases* $x > 0$ *and* $x < 0$ *separately.*

Solo problems

Problem 2. Prove the ratio test in the case where $L = \lim_{n\to\infty} |a_{n+1}/a_n| > 1$. That is, prove $\sum a_n$ diverges when $L > 1$. *Note: we worked on this in class on a worksheet, but I want you to write up your proof.*

Problem 3. Suppose that $\sum a_n$ and $\sum b_n$ are convergent series and let $A = \sum a_n$ and $B = \sum b_n$. Use partial sum sequences and the algebraic limit theorems for sequences to prove the following:

1. $\sum(a_n + b_n)$ converges to $A + B$.
2. $\sum ka_n$ converges to kA, given $k \in \mathbb{R}$.

Problem 4. Use part 1 below as a lemma for part 2.

1. Show that for any real numbers $x, y \geq 0$, the inequality $\sqrt{xy} \leq x + y$ holds.
2. Prove that if $a_n, b_n \geq 0$ for all n and $\sum a_n$ and $\sum b_n$ are convergent series, then $\sum \sqrt{a_n b_n}$ converges.

Problem 5. Read about the alternating series test in our textbook (Theorem 15.3). I want you to prove each of the series below converges. However, you may only use the alternating series test for one series. For the other series, you may use the other tests we've learned and other homework problems.

1. $\sum(-1)^n \frac{2^n}{n!}$
2. $\sum(-1)^n \frac{1}{n^{1/2}}$
3. $\sum(-1)^n \frac{1}{n^3}$
4. $\sum(-1)^n \frac{1}{n^4}$

Problem 6. In the following, unless otherwise stated, assume that $\sum a_n$ is a series that possibly contains both positive and negative terms.

1. Give an example of a divergent series $\sum a_n$ for which $\sum a_n^2$ converges.
2. Give an example of a convergent series $\sum a_n$ for which $\sum a_n^2$ diverges. *Hint: as you think about this problem, consider Problem 3 from Homework 6.*