Math 301, Fall 2021 — Homework 8

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Due November 5

Instructions. Please submit your solutions to the following problems on Gradescope. For the group part, please type your answers in LaTeX and submit the output PDF. You'll only submit one write-up for the whole group. **A small change in group work:** for redos on the group problem, I'll ask you to submit redos individually rather than as a group. For the other problems, you may handwrite solutions or use LaTeX. Make sure to select which problem is on each page in Gradescope.

Group problem

Problem 1. Use the Intermediate Value Theorem to prove the following statements.

1. Let $f(x) = x^5 - 5x^4 + 11x^3 - 12x^2 + 7x - 1$. Then f has at least one real root.
2. The equation $xe^x = 2$ has at least one solution in the interval $(0, 1)$.
3. The equation $x = \cos x$ is true for some $x \in (0, \pi/2)$.
4. Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions on the interval $[a, b]$ such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Prove that $f(x_0) = g(x_0)$ for at least one point $x_0 \in [a, b]$.

Solo problems

Problem 2. A *rational function* is a function of the form $f(x) = p(x)/q(x)$ where p and q are polynomials. Use the sequential definition of continuity and algebraic limit theorems for sequences to prove that a rational function f is continuous at any point $x_0 \in \mathbb{R}$ such that $q(x_0) \neq 0$.

Problem 3. Consider the function $f(x) = x^3$.

1. Give an ϵ-δ proof to show that f is continuous at any point $x_0 \in \mathbb{R}$. Note that in Homework 7 you were asked to give a proof in the case when $x_0 = 1$. Your proof here will be very similar, but notice that your δ will have to depend on both ϵ and a generic choice of x_0; that's ok.
2. What if we restrict x_0 to be any point in the interval $[2, 3]$? Can we find a choice of δ that depends only on ϵ and not on x_0?

Problem 4. Prove that the following functions fail to be continuous at point $x_0 = 0$.

1. $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$;

2. $g(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

3. $h(x) = \begin{cases} x/|x| & x \neq 0 \\ 0 & x = 0. \end{cases}$

Problem 5. In class we proved that if $f : [a, b] \to \mathbb{R}$ is continuous, then it achieves its maximum at some point $y_0 \in [a, b]$. Mimic that proof to show that f also achieves its minimum at some point $x_0 \in [a, b]$.