Math 301, Spring 2023 — Homework 9

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Due April 14

Instructions. Please submit your solutions to the following problems on Gradescope. Your proof answers should be written in complete sentences and avoid using symbols like $\Rightarrow$, $\therefore$, or $\because$. Edit rough drafts and reread the guidelines for writing mathematics before submitting. **Please typeset Problem 1 with LaTeX**. When you submit handwritten solutions, make sure your scan is clear, well-aligned, and as readable as possible. Make sure to select which problem is on each page in Gradescope.

Problem 1. Consider the following functions $f : D \to \mathbb{R}$ on their given domains D. Prove that each is not uniformly continuous on D.

a. $f(x) = x^3, D = \mathbb{R}$

b. $f(x) = 1/(1-x), D = (1, \infty)$

c. $f(x) = \sin(1/x), D = (0, 1)$

Problem 2. A function $f : D \to \mathbb{R}$ is called a *Lipschitz on* D if there exists $M > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M$$

for all $x, y \in D$ with $x \neq y$. Prove that if f is Lipschitz, then it's uniformly continuous on D.

Problem 3. Let $c \neq 0$. Prove that if f and g are uniformly continuous on D then $cf + g$ is uniformly continuous on D.

Problem 4. Let

$$f(x) = \begin{cases} x^n \sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Prove that$f$ is differentiable at 0 for any $n \geq 2$.

Problem 5. Let

$$f(x) = \begin{cases} x \sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Prove that f is not differentiable at 0.

Problem 6. Let $\alpha \geq 1$ be a given real number. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$|f(x) - f(y)| \leq |x - y|^\alpha$$

for all $x, y \in \mathbb{R}$.

a. Prove that if $\alpha > 1$ then f is differentiable at a for every $a \in \mathbb{R}$.

b. Give a counterexample that shows the statement above is not necessarily true if $\alpha \geq 1$. *Hint: take a look at Exercise 3.5 in Section 3 of the textbook.*