§ 2.5 Subsequences

Def Let $(a_n)_{n=1}^{\infty} = (a_1, a_2, \dots)$ be a sequence. Given

$$n_1 < n_2 < n_3 < \dots$$

a strictly increasing sequence of natural numbers, we can define a new sequence

$$(a_{n_k})_{k=1}^{\infty} = (a_{n_1}, a_{n_2}, a_{n_3}, \dots)$$

called a subsequence of (a_n).

Example Given $(a_n) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots) = (\frac{1}{n})_{n=1}^{\infty}$

(a) $(1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7})$ is a subsequence, $n_k = 2k-1$

(b) $(1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \dots)$ is a subsequence, $n_k = 10^{k-1}$

(c) $(\frac{1}{2}, 1, \frac{1}{4}, \frac{1}{3}, \frac{1}{6}, \frac{1}{5}, \dots)$ is not a subsequence

(d) $(1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \dots)$ is not a subsequence.

Theorem If (a_n) converges, then every subsequence of (a_{n_k}) converges (to the same limit).

Proof Let (a_{n_k}) be a given subsequence of (a_n).
We must show that for each $\varepsilon>0$, there exists $N \in \mathbb{N}$ such that $|a_{n_k} - L| < \varepsilon$ for all $k \geq N$ where $L = \lim_{n\to\infty} a_n$.

Let $\varepsilon > 0$. Since $a_n \to L$, there exists $N \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for all $n \geq N$. Suppose $k \geq N$. Then $n_k \geq N$. Therefore $|a_{n_k} - L| < \varepsilon$.

Example Let $a_n = (-1)^n$ for each $n \in \mathbb{N}$. Prove that (a_n) diverges.

Proof Since $(a_{2k}) = (1, 1, \dots)$ it is clear that $\lim_{k\to\infty} a_{2k} = 1$. Moreover, since $(a_{2k-1}) = (-1, -1, \dots)$ it is clear that $\lim_{k\to\infty} a_{2k-1} = -1$. Since (a_n) has subsequences that converge to different limits, it does not converge.