## § 2.5 Bolzano - Weierstrass Theorem

<u>Theorem</u> (Bolzano - Weierstrass) Every bounded sequence has a convergent subsequence.

<u>Proof</u>. Let $(a_n)$ be a bounded sequence. Then there exists $M>0$ such that $|a_n| \le M$ for all $n \in \mathbb{N}$.

Consider $[-M, 0]$ and $[0, M]$. At least one of these intervals has infinitely many terms of $(a_n)$. Suppose without loss of generality that $[0, M]$ does. Let $I_1 = [0, M]$ and let $a_{n_1}$ be the first term of $(a_n)$ that is an element of $I_1$. Consider $[0, M/2]$ and $[M/2, M]$. At least one of these has infinitely many terms of $(a_n)$. Suppose without loss of generality that $[0, M/2]$. Let $I_2 = [0, M/2]$ and let $a_{n_2}$ be the first term after $a_{n_1}$ of $(a_n)$ that is an element of $I_2$.

Proceeding inductively, we get a subsequence $(a_{n_k})$ of $(a_n)$ a collection of nested closed intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ such that $a_{n_k} \in I_k$ for all $k \in \mathbb{N}$. By the nested interval property, there exists $x \in \bigcap_{n=1}^{\infty} I_n$. We claim $(a_{n_k})$ converges to $x$. Let $\varepsilon > 0$. Since $x, a_{n_k} \in I_k$ and $\text{length}(I_k) = \frac{M}{2^{k-1}}$,

$$|a_{n_k} - x| < \frac{M}{2^{k-1}}$$

for all $k \in \mathbb{N}$. Since $\lim_{k \to \infty} \frac{M}{2^{k-1}} = 0$, there exists $N \in \mathbb{N}$ such that $\frac{M}{2^{k-1}} < \varepsilon$ for all $k \geq N$. Suppose $k \geq N$. Then

$$|a_{n_k} - x| < \frac{M}{2^{k-1}} < \varepsilon.$$