§10 Cauchy Sequences

Def A sequence (a_n) is a Cauchy sequence if $\forall\ \varepsilon>0\ \exists\ N>0$ so that $|a_n - a_m| < \varepsilon$ when $n, m > N$.

Intuition A sequence is Cauchy if its terms are eventually near each other. They "clump" together.

Lemma If (a_n) converges then it's a Cauchy sequence.

Proof Let $\varepsilon>0$ and let $L = \lim_{n\to\infty} a_n$. There exists N so that $|a_n - L| < \varepsilon/2$ when $n>N$. Suppose $n, m > N$. Then

$$
\begin{aligned}
|a_n - a_m| &= |a_n - L + L - a_m| \\
&\le |a_n - L| + |a_m - L| \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon.
\end{aligned}
$$

Lemma If (a_n) is a Cauchy sequence then it's bounded.

Proof Since (a_n) is Cauchy, there exists N so that $|a_n - a_m| < 1$ when $n, m > N$. Suppose $n > N$.

Then

$$\begin{aligned}|a_n| &= |a_n - a_{N+1} + a_{N+1}| \\ &\le |a_n - a_{N+1}| + |a_{N+1}| \\ &< 1 + |a_{N+1}|\end{aligned}$$

Let $M = \max\{|a_1|, |a_2|, \ldots, |a_N|, 1 + |a_{N+1}|\}$.

Then $|a_n| \le M$ for all $n \ge 1$.

Theorem Let $(a_n) \subseteq \mathbb{R}$ be a sequence of real numbers. Then (a_n) converges if and only if it's a Cauchy sequence.

Remark This theorem doesn't hold if (a_n) is a sequence of rational numbers. The proof will depend on Bolzano-Weierstrass, which was a consequence of Monotone Convergence, which depended on the Completeness Axiom ($\mathbb{R}$ has the least upper bound property).

Proof The forward implication was alread proved in the first lemma above. We now prove the backward implication. Suppose (a_n) is Cauchy. Then (a_n) is bounded and there exists a convergent subsequence (a_{n_k}) by the Bolzano-Weierstrass Theorem. Let $L = \lim_{k \to \infty} a_{n_k}$. We claim $\lim_{n \to \infty} a_n = L$. Let $\varepsilon > 0$. There exists N_1 so that $|a_{n_k} - L| < \varepsilon/2$ when $k > N_1$. Also, there exists N_2 so that $|a_n - a_m| < \varepsilon/2$ when $n, m > N_2$. Define $N = \max\{N_1, N_2\}$. Suppose $n > N$, $k > N$ and observe that $n_k > N$ as well and so

$$\begin{aligned}
|a_n - L| &= |a_n - a_{n_k} + a_{n_k} - L| \\
&\leq |a_n - a_{n_k}| + |a_{n_k} - L| \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon.
\end{aligned}$$