

S 10, 11 Liminf and Limsup

Goal Define generalized notions of limits that characterize the asymptotic behavior of all sequences including those that diverge by oscillating

Can we say more than "DNE" when discussing sequences like $a_n = (-1)^n$, $n \geq 1$?

Def Let (a_n) be a sequence. For each $n \geq 1$,

$$\text{let } u_n = \inf \{a_k : k > n\} = \inf \{a_{n+1}, a_{n+2}, a_{n+3}, \dots\}$$

$$v_n = \sup \{a_k : k > n\} = \sup \{a_{n+1}, a_{n+2}, a_{n+3}, \dots\}.$$

The limit inferior of (a_n) is denoted as $\liminf_{n \rightarrow \infty} a_n$

$$\text{and given by } \liminf_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} u_n$$

The limit superior of (a_n) is denoted as $\limsup_{n \rightarrow \infty} a_n$

$$\text{and given by } \limsup_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} v_n.$$

Do the limits $\lim_{n \rightarrow \infty} u_n$ and $\lim_{n \rightarrow \infty} v_n$ always exist? Why?

Problem 1. Let $a_n = (-1)^n$ and let $u_n = \inf \{a_k : k > n\}$, $v_n = \sup \{a_k : k > n\}$ for all $n \geq 1$.

- Find u_1, u_2, u_3 .
- Find v_1, v_2, v_3 .
- What do you think you can conclude about $\liminf_{n \rightarrow \infty} a_n$ and $\limsup_{n \rightarrow \infty} a_n$?

$$(a_n) = (-1, 1, -1, 1, \dots)$$

- $u_1 = -1, u_2 = -1, u_3 = -1$
- $v_1 = 1, v_2 = 1, v_3 = 1$
- $\liminf_{n \rightarrow \infty} a_n = -1, \limsup_{n \rightarrow \infty} a_n = 1$

Problem 2. Repeat Problem 1 using $(a_n) = (1, 2, 3, 1, 2, 3, 1, 2, 3, \dots)$.

- $u_1 = 1, u_2 = 1, u_3 = 1$
- $v_1 = 3, v_2 = 3, v_3 = 3$
- $\liminf_{n \rightarrow \infty} a_n = 1, \limsup_{n \rightarrow \infty} a_n = 3$

Problem 3. Repeat Problem 1 using $a_n = (-1)^n/n$ but compute the first 6 terms of (u_n) and (v_n) .

- $$(a_n) = \left(-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, \dots\right)$$
- $u_1 = -\frac{1}{3}, u_2 = -\frac{1}{3}, u_3 = -\frac{1}{5}, u_4 = -\frac{1}{5}, u_5 = -\frac{1}{7}, u_6 = -\frac{1}{7}$
 - $v_1 = \frac{1}{2}, v_2 = \frac{1}{4}, v_3 = \frac{1}{4}, v_4 = \frac{1}{6}, v_5 = \frac{1}{6}, v_6 = \frac{1}{8}$
 - $\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = 0$

Problem 4. Repeat Problem 1 using $a_n = (-1)^n + (-1)^n/n$ but compute the first 6 terms of (u_n) and (v_n) .

- $$(a_n) = \left(-2, 1 + \frac{1}{2}, -1 - \frac{1}{3}, 1 + \frac{1}{4}, -1 - \frac{1}{5}, 1 + \frac{1}{6}, \dots\right)$$
- $u_1 = -1 - \frac{1}{3}, u_2 = -1 - \frac{1}{3}, u_3 = -1 - \frac{1}{5}, u_4 = -1 - \frac{1}{5}, u_5 = -1 - \frac{1}{7}, u_6 = -1 - \frac{1}{7}$
 - $v_1 = 1 + \frac{1}{2}, v_2 = 1 + \frac{1}{4}, v_3 = 1 + \frac{1}{4}, v_4 = 1 + \frac{1}{6}, v_5 = 1 + \frac{1}{6}, v_6 = 1 + \frac{1}{8}$
 - $\liminf_{n \rightarrow \infty} a_n = -1, \limsup_{n \rightarrow \infty} a_n = 1$

Problem 5. Suppose $A \subseteq B$. How do $\inf A$ and $\inf B$ compare, ie. which is bigger? What about $\sup A$ and $\sup B$?

$$\begin{aligned} \inf A &\geq \inf B & (\text{try proving these}) \\ \sup A &\leq \sup B \end{aligned}$$

Problem 6. Consider the sequences (u_n) and (v_n) defined above.

- a. Are they increasing or decreasing?
- b. Why do they converge when (a_n) is bounded?
- c. What if (a_n) is bounded above but not bounded below?
- d. What if (a_n) is bounded below but not bounded above?

① Let $n \geq 1$. Note $\{a_k : k > n+1\} \subseteq \{a_k : k > n\}$.

Therefore $u_{n+1} \geq u_n$ and $v_{n+1} \leq v_n$ by Problem 5.

So (u_n) is increasing and (v_n) is decreasing.

② When (a_n) is bounded (ie. $\exists M > 0, \forall n \geq 1, |a_n| \leq M$)

(u_n) and (v_n) are bounded (try proving this).

Therefore, by Monotone Convergence Theorem they converge.

And so $\liminf_{n \rightarrow \infty} a_n$ and $\limsup_{n \rightarrow \infty} a_n$ exist (and are finite)

③ If (a_n) is bounded above but not bounded below,

$u_n = -\infty$ for all $n \geq 1$ and So $\liminf_{n \rightarrow \infty} a_n = -\infty$,

But we can only conclude $\limsup_{n \rightarrow \infty} a_n < \infty$ (it might be $-\infty$)

④ If (a_n) is bounded below but not bounded above,

$v_n = +\infty$ for all $n \geq 1$, so $\limsup_{n \rightarrow \infty} a_n = +\infty$

But we can only conclude $\liminf_{n \rightarrow \infty} a_n > -\infty$ (it might

be $+\infty$)

Theorem Let (a_n) be a given sequence and let

$-\infty \leq L \leq \infty$. Then $\lim_{n \rightarrow \infty} a_n = L$ if and only if

$$\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = L.$$

We'll prove the case when

$L \neq \pm\infty$.

Proof (\Rightarrow) Assume (a_n) converges to L . Then

(a_n) is bounded and (u_n) and (v_n) converge by monotone convergence theorem. We claim

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} v_n = L. \text{ Let } \varepsilon > 0. \text{ There exists } N$$

so that $|a_n - L| < \varepsilon$ for all $n > N$.

Note then that $a_n > L - \varepsilon$ for all $n > N$. Therefore

$L - \varepsilon$ is lower bound for the set $\{a_n : n > N\}$,

$$\text{which implies } L - \varepsilon \leq \inf \{a_n : n > N\}$$

$$= u_N$$

$$\leq v_n$$

for all $n > N$. Thus $L - \varepsilon \leq \lim_{n \rightarrow \infty} u_n$ by the

Order Limit Theorem. Since ε was arbitrary, $L \leq \lim_{n \rightarrow \infty} u_n$.

A similar argument yields $\lim_{n \rightarrow \infty} v_n \leq L$. Therefore

$$L \leq \lim_{n \rightarrow \infty} u_n \leq \lim_{n \rightarrow \infty} v_n \leq L \text{ which implies } \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} v_n = L.$$

(\Leftarrow) Assume $\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = L$. We aim to

prove (a_n) converges to L . For every $n \geq 2$

$$u_{n-1} \leq a_n \leq v_{n-1}.$$

By the Squeeze Theorem (a_n) converges to L .

Lemma (Squeeze Theorem) Let (a_n) , (b_n) , (c_n) be sequences such that $a_n \leq b_n \leq c_n$ for all but finitely many n . If $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$ then (b_n) converges to L .

Proof Let $\epsilon > 0$. There exists N_1 such that $|a_n - L| < \epsilon$ for all $n > N_1$, N_2 such that $|c_n - L| < \epsilon$ for all $n > N_2$, and N_3 such that $a_n \leq b_n \leq c_n$ for all $n > N_3$. Let $N = \max\{N_1, N_2, N_3\}$ and suppose $n > N$. Then

$$L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon$$

which implies $|b_n - L| < \epsilon$.