§ 2.7 Properties of Infinite Series

Recap on infinite series: $\sum_{k=1}^{\infty} a_k$

- (a_n) is the sequence of terms
- (s_n) where $s_n = \sum_{k=1}^{n} a_k$ is the sequence of partial sums
- $\sum_{k=1}^{\infty} a_k$ converges to A or $\sum_{k=1}^{\infty} a_k = A$ means

  $\lim_{n\to\infty} s_n = A$.
- $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

  (in fact $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$).

Theorem (Algebraic Limit Theorem for Series)

If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, then

(i) $\sum_{k=1}^{\infty} ca_k = cA$ for any $c \in \mathbb{R}$,

(ii) $\sum_{k=1}^{\infty} (a_k + b_k) = A + B$.

Proof of (i) Let $s_n = \sum_{k=1}^{n} a_k$ and let $t_n = \sum_{k=1}^{n} ca_k$ for each $n \in \mathbb{N}$. Then since $s_n \to A$ and $t_n = cs_n$, by the Algebraic Limit Theorem (i) for sequences $t_n \to cA$.

Theorem (Cauchy Criterion for Series) The series $\sum_{k=1}^{\infty} a_k$ converges if and only if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > m \geq N$,

$$|a_{m+1} + \cdots + a_n| < \varepsilon.$$

Proof Let (s_n) be the partial sum sequence of $\sum_{k=1}^{\infty} a_k$. Note that $\sum_{k=1}^{\infty} a_k$ converges if and only if (s_n) converges. Further (s_n) converges if and only if (s_n) is Cauchy. Finally (s_n) is Cauchy if and only if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|s_n - s_m| < \varepsilon$ for all $n, m \geq N$. Notice, when $n > m$

$$|s_n - s_m| = |(a_1 + \cdots + a_m + \cdots + a_n) - (a_1 + \cdots + a_m)|$$

$$= |a_{m+1} + \cdots + a_n|.$$

Theorem (nth term test) If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k \to \infty} a_k = 0$. Contrapositive: If $\lim_{k \to \infty} a_k \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges.

Proof Let $\varepsilon > 0$. Since $\sum_{k=1}^{\infty} a_k$ converges, there exists $N \in \mathbb{N}$ such that for all $n > m \geq N$, $|a_{m+1} + \cdots + a_n| < \varepsilon$. In particular, suppose $m \geq N$ and $n = m+1$. Then $|a_{m+1}| < \varepsilon$. Therefore, for all $n \geq N+1$, $|a_n| < \varepsilon$.

Theorem (Comparison test) Let (a_n) and (b_n) be sequences where $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$. (actually this condition can be loosened to there exists $N \in \mathbb{N}$ such that $0 \leq a_n \leq b_n$ for all $n \geq N$) Then

(i) if $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

(ii) if $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges

Proof of (i) We must show, by the Cauchy criterion for series, that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_{m+1} + \cdots + a_n| < \varepsilon$ whenever $n > m \geq N$. Let $\varepsilon > 0$.

Since $\sum_{k=1}^{\infty} b_k$ converges, there exists $N \in \mathbb{N}$ such that $|b_{m+1} + \cdots + b_m| < \varepsilon$ for all $n > m \geq N$. Suppose $n > m \geq N$.

Then

$$\begin{aligned} |a_{m+1} + \cdots + a_n| &= a_{m+1} + \cdots + a_n \\ &\leq b_{m+1} + \cdots + b_n \\ &= |b_{m+1} + \cdots + b_n| \\ &< \varepsilon. \end{aligned}$$