

§14 Series

Def A series is an infinite sum of the terms of a sequence $(a_n)_{n=1}^{\infty}$:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

The partial sum sequence $(s_n)_{n=1}^{\infty}$ is the sequence

$$s_1 = a_1, \quad s_2 = a_1 + a_2, \quad s_3 = a_1 + a_2 + a_3, \dots$$

The series $\sum_{n=1}^{\infty} a_n = A$ (ie. $\sum_{n=1}^{\infty} a_n$ converges to A)

if $\lim_{n \rightarrow \infty} s_n = A$ (the partial sum sequence converges to A).

Examples ① $\sum_{n=1}^{\infty} \frac{1}{n}$ (harmonic series) *diverges*

② $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges when $p > 1$*

③ $\sum_{n=1}^{\infty} r^n$ (geometric series) *converges when $|r| < 1$.*

④ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ *converges when $p > 0$*

Example $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

Proof Let $s_n = \sum_{k=1}^n \frac{1}{k}$. Notice

$$s_1 = 1$$

$$s_2 = 1 + \frac{1}{2}$$

$$s_3 = 1 + \frac{1}{2} + \frac{1}{3}$$

$$s_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{2}{2}$$

$$s_8 = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\frac{1}{2}} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 1 + \frac{3}{2}$$

$$s_{16} > 1 + \frac{4}{2}$$

and in general $s_{2^k} > 1 + \frac{k}{2}$ for all $k \geq 1$.

Therefore $\lim_{k \rightarrow \infty} s_{2^k} = \infty$ and $\lim_{n \rightarrow \infty} s_n = \infty$ since

any increasing sequence with a subsequence that diverges to ∞ must itself diverge to ∞ (try proving this).

Example $\sum_{k=1}^{\infty} r^k$ converges if and only if $|r| < 1$.

Let $S_n = \sum_{k=1}^n r^k$ for any $n \geq 1$.

$$\begin{aligned} \text{Then } S_n - rS_n &= \sum_{k=1}^n r^k - r \sum_{k=1}^n r^k \\ &= (r + r^2 + \dots + r^n) - r(r + r^2 + \dots + r^n) \\ &= r - r^{n+1} \end{aligned}$$

$$\text{so } S_n = \frac{r - r^{n+1}}{1 - r} \text{ for any } n \geq 1.$$

Since (r^{n+1}) converges (to 0) if and only if $|r| < 1$, it follows (S_n) converges (to $\frac{r}{1-r}$) if and only if $|r| < 1$.

Theorem (Algebraic Limit Theorems) Suppose $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$. Then

$$\textcircled{1} \sum_{k=1}^{\infty} k a_k = kA \text{ for any } k \in \mathbb{R}$$

$$\textcircled{2} \sum_{k=1}^{\infty} (a_k + b_k) = A + B.$$

Proof Exercise.

Theorem (Cauchy criterion) $\sum_{n=1}^{\infty} a_n$ converges if and only

if $\forall \epsilon > 0 \exists N > 0$ such that $\left| \sum_{k=m+1}^n a_k \right| < \epsilon$

when $n > m > N$.

We'll prove the forward implication and leave the backward implication as an exercise.

Proof Suppose $\sum_{n=1}^{\infty} a_n$ converges. Then

the sequence (s_n) of partial sums converges, and so is a Cauchy sequence. Let $\epsilon > 0$.

Then $\exists N$ such that $|s_n - s_m| < \epsilon$ for all $n, m > N$. Suppose $n > m > N$. Then

$$\left| \sum_{k=m+1}^n a_k \right| = |s_n - s_m| < \epsilon.$$

Theorem (Test for Divergence aka n th term test)

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$.

Proof Let $\epsilon > 0$. Since $\sum_{n=1}^{\infty} a_n$ converges, $\exists N$

so that $\left| \sum_{k=m+1}^n a_k \right| < \epsilon$ when $n > m > N$. Suppose

$m > N$. Then, with $n = m+1$, $|a_m| = \left| \sum_{k=m+1}^{m+1} a_k \right| < \epsilon$.

Thus (a_n) converges to 0.