

## S 14 Series

---

Theorem (Test for Divergence aka nth term test)

If  $\sum_{n=1}^{\infty} a_n$  converges, then  $\lim_{n \rightarrow \infty} a_n = 0$ .

Proof Let  $\varepsilon > 0$ . Since  $\sum_{n=1}^{\infty} a_n$  converges,  $\exists N$

so that  $\left| \sum_{k=m+1}^n a_k \right| < \varepsilon$  when  $n > m > N$ . Suppose

$m > N$ . Then, with  $n = m+1$ ,  $|a_m| = \left| \sum_{k=m+1}^{m+1} a_k \right| < \varepsilon$ .

Thus  $(a_n)$  converges to 0.

Theorem (Comparison Test) Let  $\sum a_n$  be a

given series and suppose  $a_n > 0$  for all  $n$ . Then

① if  $\sum a_n$  converges and  $|b_n| \leq a_n$  for all

(but finitely many)  $n$ , then  $\sum b_n$  converges.

② if  $\sum a_n = +\infty$  (ie. diverges) and  $b_n \geq a_n$

for all (but finitely many)  $n$ , then  $\sum b_n$  diverges.

Proof ① We'll show  $\sum b_n$  converges by showing it satisfies the Cauchy criterion. Let  $\epsilon > 0$ . Since  $\sum a_n$  converges,  $\exists N$  so that  $\left| \sum_{k=m+1}^n a_k \right| < \epsilon$  when  $n > m > N$ . Suppose  $n > m > N$ . Then

$$\begin{aligned} \left| \sum_{k=m+1}^n b_k \right| &\leq \sum_{k=m+1}^n |b_k| \\ &\leq \sum_{k=m+1}^n a_k \\ &= \left| \sum_{k=m+1}^n a_k \right| \\ &< \epsilon \end{aligned}$$

② We'll show  $\sum b_n$  diverges by showing its sequence of partial sums diverges to  $\infty$ .

Let  $s_n = \sum_{k=1}^n b_k$  and  $t_n = \sum_{k=1}^{\infty} a_k$  for all  $n \geq 1$ .

Then  $s_n \geq t_n$  for all  $n \geq 1$ . Let  $M > 0$ . Since  $\lim_{n \rightarrow \infty} t_n = \infty$ ,  $\exists N$  such that  $t_n > M \quad \forall n > N$ .

Suppose  $n > N$ . Then  $s_n \geq t_n > M$ , so  $\lim_{n \rightarrow \infty} s_n = \infty$ .

**Problem 1.** For each of the following series, decide whether it converges, giving a justification based on one of the tests we've introduced along with the convergence of our baseline examples (geometric series and  $p$ -series). These are the warm-up problems

- a.  $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2}$
- b.  $\sum_{n=1}^{\infty} \frac{1}{2^n + n}$
- c.  $\sum_{n=2}^{\infty} \frac{1}{\ln n}$
- d.  $\sum_{n=1}^{\infty} \frac{n}{n+1}$
- e.  $\sum_{n=2}^{\infty} \frac{n^2}{n^3 - 1}$

- (a) converges by comparison with  $\sum \frac{1}{n^2}$
- (b) converges by comparison with  $\sum \frac{1}{2^n}$
- (c) diverges by comparison with  $\sum \frac{1}{n}$
- (d) diverges by Test for Divergence
- (e) diverges by comparison with  $\sum \frac{1}{n}$

**Problem 2.** Take as given that  $p$ -series converge when  $p > 1$  and alternating  $p$ -series converge for all  $p > 0$ . Make a conjecture about whether each of the following statements is true or false and give a brief explanation or counterexample as justification.

- a. If  $(a_n)$  is a Cauchy sequence, then the series  $\sum a_n$  converges.
- b. If the series  $\sum a_n$  converges, then  $(a_n)$  is a Cauchy sequence.
- c. If the series  $\sum a_n$  converges, then the series  $\sum |a_n|$  converges.
- d. If the series  $\sum |a_n|$  converges, then the series  $\sum a_n$  converges.

- (a) false,  $a_n = \frac{1}{n}$  is a counterexample
- (b) true  $\sum a_n$  converges  $\Rightarrow \lim a_n = 0 \Rightarrow (a_n)$  converges  
 $\Rightarrow (a_n)$  Cauchy
- (c) false,  $a_n = \frac{(-1)^{n+1}}{n}$  is a counterexample
- (d) true, see below

Theorem (Absolute convergence test) If  $\sum |a_n|$  converges  
then  $\sum a_n$  converges.

Proof Let  $\epsilon > 0$ . Since  $\sum |a_n|$  converges  $\exists N$   
such that  $\left| \sum_{k=m+1}^n |a_k| \right| < \epsilon$  when  $n > m > N$ . Suppose  
 $n > m > N$ . Then  $\left| \sum_{k=m+1}^n a_k \right| \leq \sum_{k=m+1}^n |a_k|$   
 $= \left| \sum_{k=m+1}^n |a_k| \right|$   
 $< \epsilon$

Therefore  $\sum a_n$  satisfies the Cauchy criterion and  
so converges.