§14 Series

Theorem (Ratio test) Let $\sum a_n$ be a given series of non-zero terms and let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

Then ① if $L<1$, $\sum a_n$ converges

② if $L>1$, $\sum a_n$ diverges

③ if $L=1$, the test is inconclusive

Proof ① Suppose $L<1$. Then $\exists\ \varepsilon_0>0$ such that $L+\varepsilon_0<1$. Further, $\exists\ N>0$ such that

$$\left|\, \left|\frac{a_{n+1}}{a_n}\right| - L \,\right| < \varepsilon_0$$

for all $n>N$. Thus $\left|\frac{a_{n+1}}{a_n}\right| < L+\varepsilon_0$ for all $n>N$.

Suppose $n>N$. Then

$$\begin{aligned} |a_n| &= \left|\frac{a_n}{a_{n-1}}\frac{a_{n-1}}{a_{n-2}}\cdots\frac{a_{N+2}}{a_{N+1}}a_{N+1}\right| \\ &= \left|\frac{a_n}{a_{n-1}}\right|\cdots\left|\frac{a_{N+2}}{a_{N+1}}\right| |a_{N+1}| \\ &< (L+\varepsilon_0)^{n-(N+2)+1}|a_{N+1}| \\ &= C(L+\varepsilon_0)^n \end{aligned}$$

where $C = (L+\varepsilon_0)^{-(N+1)}|a_{N+1}|$. Since $L+\varepsilon_0<1$,

$\sum_{n=N+1}^{\infty} C(L+\varepsilon_0)^n$ is a convergent geometric series.

Therefore by the comparison test, $\sum a_n$ converges.

Problem 1. Determine whether the following statement is true or false; justify or give a counterexample. *If* $\sum |a_n|$ *diverges, then* $\sum a_n$ *diverges.*

False, $a_n = \frac{(-1)^{n+1}}{n}$ is a counterexample

Problem 2. Consider the series $\sum a_n$, let $L = \lim_{n\to\infty} |a_{n+1}/a_n|$. Let's try to prove the second statement of the ratio test: *if* $L > 1$, *then* $\sum a_n$ *diverges.*

a. Adapt the steps of the proof for the case $L < 1$ to the case $L > 1$.

b. What goes wrong with your adpatation of the proof if you try using the comparison test to compare with a divergent geometric series? Where does the logic fail?

c. Fix your proof using the Test for Divergence.

(a) Let $L > 1$. Then $\exists$ $\varepsilon_0 > 0$ such that $L - \varepsilon_0 > 1$. Moreover, $\exists$ $N > 0$ such that

$$\left| \left| \frac{a_{n+1}}{a_n} \right| - L \right| < \varepsilon_0$$

for all $n > N$. Therefore, $\left| \frac{a_{n+1}}{a_n} \right| > L - \varepsilon_0$ for all $n > N$. Suppose $n > N$. Observe that

$$|a_n| = \left| \frac{a_n}{a_{n-1}} \cdots \frac{a_{N+2}}{a_{N+1}} a_{N+1} \right|$$

$$> (L - \varepsilon_0)^{n-(N+2)+1} |a_{N+1}|$$

Therefore, since $L - \varepsilon_0 > 1$, $\lim_{n\to\infty} |a_n| = +\infty$, which implies $\lim_{n\to\infty} |a_n| \neq 0$, which implies $\lim_{n\to\infty} a_n \neq 0$. By the test for divergence, $\sum a_n$ diverges.

Problem 3. Let's try to prove the third statement of the ratio test: *if $L = 1$, then the ratio test is inconclusive.*

a. Find a convergent series where $L = 1$.

b. Find a divergent series where $L = 1$.

c. Explain why your two examples prove the third statement of the ratio test.

③ Consider $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$. For both of these series, $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$. Therefore since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we see either outcome is possible when $L = 1$.

Problem 4. Explain whether the following series converge.

a. $\sum_{n=1}^{\infty} \frac{n^5}{7^n}$

b. $\sum_{n=1}^{\infty} \frac{n^3}{n!}$

c. $\sum_{n=1}^{\infty} \frac{n!}{2^n}$

ⓐ converges : $L = \lim_{n\to\infty} \frac{(n+1)^5}{7^{n+1}} \cdot \frac{7^n}{n^5} = \frac{1}{7} < 1$

ⓑ converges : $L = \lim_{n\to\infty} \frac{(n+1)^3}{(n+1)!} \frac{n!}{n^3} = \lim_{n\to\infty} \frac{(n+1)^3}{(n+1)n^3} = 0 < 1$

ⓒ diverges : $L = \lim_{n\to\infty} \frac{(n+1)!}{2^{n+1}} \frac{2^n}{n!} = \lim_{n\to\infty} \frac{n+1}{2} = \infty > 1$