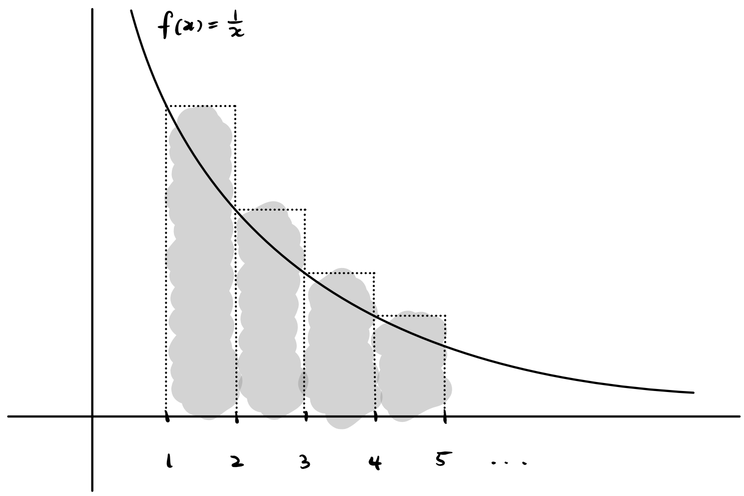
§15 Integral Test

Let's give a new proof that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.



Observe that

$$\sum_{n=1}^{\infty} \frac{1}{n} = \text{sum of rectangles in picture above}$$

$$> \int_1^{\infty} \frac{1}{x}\,dx$$

$$= \lim_{b \to \infty} \int_1^b \frac{1}{x}\,dx$$

$$= \lim_{b \to \infty} \ln x \Big|_1^b$$

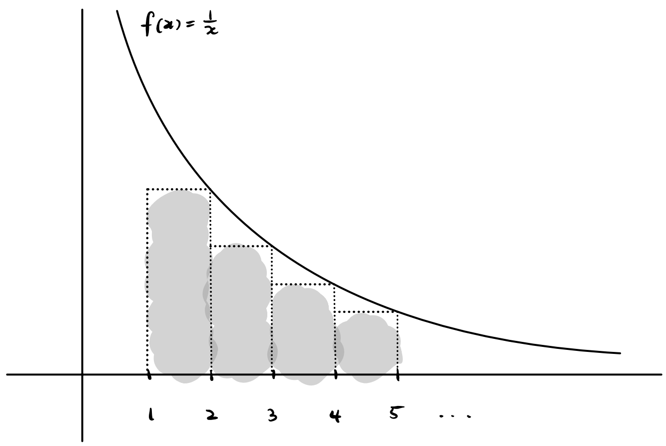
$$= \lim_{b \to \infty} \ln b = +\infty.$$

Problem 1. Let's try to prove $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

a. Sketch a graph of the function $f(x) = 1/x^2$.

b. Relate the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ to the area under the graph of $f(x)$ over the interval $[1, \infty)$. *Be careful here; it's slightly more subtle than our first example.*

c. Compute $\int_1^{\infty} f(x)\,dx$ and make a conclusion.



Observe that

$$\sum_{n=2}^{\infty} \frac{1}{n^2} = \text{sum of the rectangles above}$$

$$< \int_1^{\infty} \frac{1}{x^2}\,dx$$

$$= \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$$

$$= \lim_{b\to\infty} -x^{-1}\Big|_1^b$$

$$= \lim_{b\to\infty} 1 - \frac{1}{b}$$

$$= 1$$

More formally, we see then that the partial sum sequence (s_n) is monotone and bounded above by 2, and so converges by Monotone Convergence Thm.

Problem 2. The *Integral Test* states that if f is positive and decreasing on the interval $[1, \infty)$, $\lim_{x\to\infty} f(x) = 0$, and $a_n = f(n)$ for all $n \geq 1$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\,dx$ is finite.

a. Compute $\int_1^{\infty} \frac{1}{x^p}\,dx$ for any constant $p > 0$. Your answer should be a formula in terms of p.

b. Use the Integral Test to find the values of $p > 0$ for which the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

(a)

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx$$

$$= \lim_{b\to\infty} \frac{1}{-p+1} x^{-p+1}\Big|_1^b \quad \text{when } p \neq 1$$

$$= \lim_{b\to\infty} \frac{1}{p-1}\left(1 - \frac{1}{b^{p-1}}\right)$$

$$= \begin{cases} \frac{1}{p-1} & \text{when } p > 1 \\ +\infty & \text{when } p < 1 \end{cases}$$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.