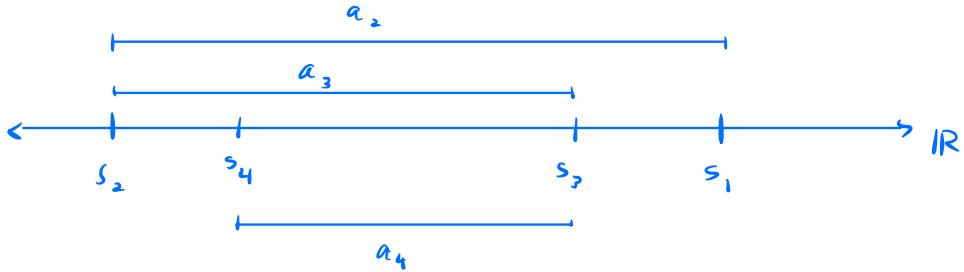


## § 15 Alternating Series

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Theorem (Alternating Series Test) Suppose  $(a_n)$  is positive, decreasing and  $\lim_{n \rightarrow \infty} a_n = 0$ . Then  $\sum (-1)^n a_n$  converges to a value A.

$$\text{Moreover, } \left| \sum_{k=1}^n (-1)^{k+1} a_k - A \right| \leq a_n$$



**Problem 1.** Suppose  $(a_n)$  is a decreasing sequence of positive numbers that converges to 0. Let  $(s_n)$  be the partial sum sequence of  $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ . Prove the following claims.

- a. The subsequence  $(s_{2k-1})$  is decreasing.

Let  $k \geq 1$ . Then

$$\begin{aligned} s_{2(k+1)-1} - s_{2k-1} &= (a_1 - a_2 + a_3 - a_4 + \dots + a_{2k+1}) - (a_1 - a_2 + \dots + a_{2k-1}) \\ &= a_{2k+1} - a_{2k} \\ &< 0 \end{aligned}$$

- b. The subsequence  $(s_{2k})$  is increasing.

Similar to a.

- c. The two subsequences above converge.

Notice  $(s_{2k-1})$  is bounded below by  $s_2$  since  $\forall k \geq 1$

$$s_{2k-1} - s_{2k} = a_{2k} > 0, \text{ which implies } s_{2k-1} > s_{2k} > s_2.$$

Similarly  $(s_{2k})$  is bounded above by  $s_1$ . Therefore each of these sequences converges by Monotone Convergence Theorem.

d.  $\lim_{k \rightarrow \infty} (s_{2k+1} - s_{2k}) = 0$ .

$$\lim_{k \rightarrow \infty} (s_{2k-1} - s_{2k}) = \lim_{k \rightarrow \infty} a_{2k} = 0$$

e. The two subsequences above converge to the same value  $A$ .

Let  $A = \lim_{k \rightarrow \infty} s_{2k-1}$  and  $B = \lim_{k \rightarrow \infty} s_{2k}$ . By the Algebraic Limit Theorem,  $0 = \lim_{k \rightarrow \infty} (s_{2k-1} - s_{2k}) = A - B$ , so  $A = B$ .

f. The sequence  $(s_n)$  converges to  $A$ .

Let  $\varepsilon > 0$ . Then there exists  $N_1, N_2 > 0$  so that

$$|s_{2k-1} - A| < \varepsilon \text{ when } k > N_1 \text{ and } |s_{2k} - A| < \varepsilon$$

when  $k > N_2$ . Let  $N = \max\{2N_1 + 1, 2N_2\}$  and .

suppose  $n > N$ . If  $n$  is odd, then  $n = 2k + 1$  for some

$k \geq 0$ . Therefore  $n > N \geq 2N_1 + 1$ , which implies  $k > N_1$ ,

and so  $|s_n - A| = |s_{2k+1} - A| < \varepsilon$ . A similar argument

holds when  $n$  is even.

g.  $|s_n - A| \leq a_n$

Observe that  $|s_n - A| \leq |s_n - s_{n-1}| = a_n$

This is because if  $n$  is even then  $s_n \leq A \leq s_{n-1}$

and if  $n$  is odd then  $s_n \geq A \geq s_{n-1}$ .

**Problem 2.** Consider the series  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$ .

- Explain why the series converges.
- Let  $L$  be the value the series converges to. Suppose we try to approximate  $L$  using the sum of the first 10 terms of the series. Give a bound on the error of this approximation.
- Suppose we want to approximate  $L$  by summing the first  $n$  terms of the series. How big should  $n$  be so that the error of this approximation is no more than 0.0001?

(a) By alternating series (notice  $a_n = \frac{1}{\sqrt{n}}$  is positive, decreasing, and converges to 0).

$$(b) a_n = \frac{1}{\sqrt{n}}$$

$$(c) |s_n - A| \leq a_n = \frac{1}{\sqrt{n}}, \text{ so } |s_n - A| < 0.0001$$

$$\text{if } \frac{1}{\sqrt{n}} < 0.0001 \iff n > \left(\frac{1}{0.0001}\right)^2 = 10^8$$

Proof Let  $s_n = \sum_{k=1}^n (-1)^{k+1} a_k$ ,  $\forall n \geq 1$ . We claim  $(s_{2n})$  and  $(s_{2n+1})$  are convergent subsequences of  $(s_n)$ .

Note  $(s_{2n})$  is increasing since

$$s_{2n} - s_{2n+2} = a_{2n+1} - a_{2n+2} \geq 0$$

Since  $(a_n)$  is decreasing, and so  $s_{2n} \geq s_{2n+2}$ .

Similarly  $(s_{2n+1})$  is decreasing. We claim  $(s_{2n})$  is bounded above by  $a_1$ . Let  $n \geq 1$ . Then

$$s_{2n} \leq s_{2n+1} \text{ since } s_{2n+1} - s_{2n} = a_{2n+1} \geq 0.$$

Therefore, since  $(s_{2n+1})$  is decreasing,  $s_{2n+1} \leq s_1 = a_1$ , and so  $s_{2n} \leq a_1$ . We claim moreover that

$(s_{2n+1})$  is bounded below by  $s_2$ . Indeed,

$$s_{2n+1} \geq s_{2n} \quad (\text{shown above})$$

$$\geq s_2 \quad (\text{since } (s_{2n}) \text{ is increasing})$$

Therefore, by the Monotone Convergence Theorem,

$(s_{2n})$  and  $(s_{2n+1})$  converge, say to  $L_1$  and  $L_2$ .

We claim  $L_1 = L_2$ . Indeed

$$\begin{aligned}
L_2 - L_1 &= \lim_{n \rightarrow \infty} s_{2n+1} - \lim_{n \rightarrow \infty} s_{2n} \\
&= \lim_{n \rightarrow \infty} (s_{2n+1} - s_{2n}) \\
&= \lim_{n \rightarrow \infty} a_{2n+1} \\
&= 0.
\end{aligned}$$

Finally we claim  $\lim_{m \rightarrow \infty} s_m = L_1$ . Let  $\varepsilon > 0$ .

Then  $\exists N_1$  and  $N_2$  such that

$$|s_{2n} - L_1| < \varepsilon \text{ when } n > N_1 \text{ and } |s_{2n+1} - L_1| < \varepsilon$$

when  $n > N_2$ . Let  $N = \max\{2N_1, 2N_2 + 1\}$ . Suppose

$m > N$ . If  $m$  is even, then  $m = 2n$  for some  $n$ , and  $m > N$

implies  $2n > N \geq 2N_1$ , and so  $n > N_1$ . Therefore

$$|a_m - L_1| = |a_{2n} - L_1| < \varepsilon. \text{ If } m \text{ is odd, then}$$

$m = 2n+1$  for some  $n$ . Then  $m > N$  implies

$2n+1 > N \geq 2N_2 + 1$ , and so  $n > N_2$ . Thus

$$|a_m - L_1| = |a_{2n+1} - L_1| < \varepsilon.$$