§ 3.2 Open and Closed Sets

Example Let $A = \{ \frac{1}{n} : n \in \mathbb{N} \}$. Then

① 0 is a limit point of A

② A is not closed

③ every element of A is an isolated point

① Let $a_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $(a_n) \subseteq A \setminus \{0\}$

and $a_n \to 0$. Therefore 0 is a limit point.

② Since $0 \notin A$, A does not contain all its limit points.

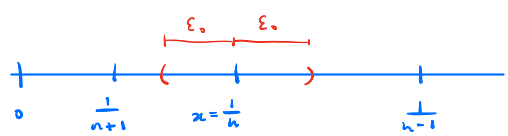
Therefore A is not closed.

③ Let $n \in \mathbb{N}$. We claim $\frac{1}{n}$ is not a limit point.

We must show $\exists$ $\varepsilon_0 > 0$ such that $V_{\varepsilon_0}(\frac{1}{n}) \cap A \setminus \{\frac{1}{n}\}$

is empty. Let $\varepsilon_0 = \dfrac{\frac{1}{n} - \frac{1}{n+1}}{2}$. Then

$$V_{\varepsilon_0}\left(\tfrac{1}{n}\right) \cap A \setminus \left\{\tfrac{1}{n}\right\} = \emptyset .$$



Example Consider $\mathbb{Q}$. Then $\mathbb{R}$ is its set of limit points.

Let $x \in \mathbb{R}$ and let $\varepsilon > 0$. By the density of $\mathbb{Q}$ in $\mathbb{R}$,

there exist $q_1 \in (x - \varepsilon, x) \cap \mathbb{Q}$ and $q_2 \in (x, x+\varepsilon) \cap \mathbb{Q}$.

Therefore $V_\varepsilon(x) \cap \mathbb{Q} \setminus \{x\}$ is non-empty. So x is a limit point.

Theorem A set $A \subseteq \mathbb{R}$ is open if and only if A^c is closed. Likewise $B \subseteq \mathbb{R}$ is closed if and only if B^c is open.

Remark There are sets which are neither open nor closed. For example $[a, b)$, where $a, b \in \mathbb{R}$ with $a < b$.

Proof Suppose A is open. To show A^c is closed we must show A^c contains all its limit points. Let x be a limit point of A^c. Then there exists a sequence $(x_n) \subseteq A^c \setminus \{x\}$ such that $x_n \to x$. Suppose $x \in A$. Then there exists $\varepsilon_0 > 0$ such that $V_{\varepsilon_0}(x) \subseteq A$. Moreover, there exists $N \in \mathbb{N}$ such that $x_n \in V_{\varepsilon_0}(x)$ for all $n \geq N$. However this is a contradiction since $x_n \in A^c$ for all $n \in \mathbb{N}$. Therefore $x \in A^c$.

Suppose A^c is closed. To show A is open we must show that for every $x \in A$ there exists $\varepsilon > 0$ such that $V_\varepsilon(x) \subseteq A$. Let $x \in A$. Since A^c contains all its limit points and $x \notin A^c$ it must be that x is not a limit point of A^c. Therefore there exists $\varepsilon_0 > 0$ such that $V_{\varepsilon_0}(x) \cap A^c \setminus \{x\} = \phi$. This means $V_{\varepsilon_0}(x) \subseteq A$.

The second statement follows from the first.

Theorem (i) The union of a finite collection of closed sets is closed.

(ii) The intersection of an arbitrary collection of closed sets is closed.

Proof (i) Let $A_1, \ldots, A_n$ be closed sets. Then $A_1^c, \ldots, A_n^c$ are open and so $\bigcap_{i=1}^{n} A_i^c$ is open.

Notice $\bigcap_{i=1}^{n} A_i^c = \left(\bigcup_{i=1}^{n} A_i \right)^c$ by De Morgan's laws. Therefore $\bigcup_{i=1}^{n} A_i$ is closed.

(ii) Let $\{A_\alpha : \alpha \in I\}$ be an arbitrary collection of closed sets where I is a given index set.

Then $\{A_\alpha^c : \alpha \in I\}$ is a collection of open sets and $\bigcup_{\alpha \in I} A_\alpha^c$ is open. Since $\bigcup_{\alpha \in I} A_\alpha^c = \left(\bigcap_{\alpha \in I} A_\alpha \right)^c$ we have that $\bigcap_{\alpha \in I} A_\alpha$ is closed.