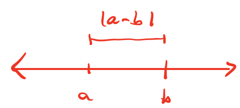
1.2 Some Preliminaries

Definition The absolute value function is given by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

for any $x \in \mathbb{R}$. The Euclidean distance function on $\mathbb{R}$ is the function $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ given by

$$d(a,b) = |a-b|.$$



Theorem Let $a, b, c \in \mathbb{R}$. Then

1. $|a| \geq 0$
2. $|ab| = |a||b|$
3. $|a+b| \leq |a| + |b|$
4. $d(a,c) \leq d(a,b) + d(b,c)$

both called the triangle inequality

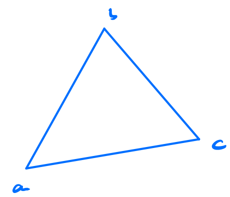
We'll prove 3 and 4.

3 Observe that

$$\begin{aligned} |a+b|^2 &= (a+b)^2 \\ &= a^2 + 2ab + b^2 \\ &= |a|^2 + 2ab + |b|^2 \\ &\leq |a|^2 + 2|a||b| + |b|^2 \\ &= (|a| + |b|)^2 \end{aligned}$$

and take the square root of both sides.

(4) Observe that



$$
\begin{aligned}
d(a,c) &= |a-c| \\
&= |a-b+b-c| \\
&\le |a-b| + |b-c| \\
&= d(a,b) + d(b,c).
\end{aligned}
$$

Theorem Let $a, b \in \mathbb{R}$. Then $a = b$ if and only if $|a-b| < \varepsilon$ for every $\varepsilon > 0$.

Proof ($\Longrightarrow$) We'll assume $a=b$ and prove that $|a-b| < \varepsilon$ for every $\varepsilon > 0$. Let $\varepsilon > 0$. Then

$$|a-b| = 0 < \varepsilon.$$

($\Longleftarrow$) We'll assume $|a-b| < \varepsilon$ for every $\varepsilon > 0$ and prove $a = b$ by contradiction. Suppose $a \neq b$ and without loss of generality assume $a > b$. Then $\varepsilon_0 = \frac{a-b}{2} > 0$. Therefore

$$
\begin{aligned}
a-b &= |a-b| \\
&< \varepsilon_0 \\
&= \frac{a-b}{2}, \text{ which is a contradiction}
\end{aligned}
$$

since it implies $2(a-b) < a-b \Rightarrow 2 < 1$.

Problem 1. Let $a, b, c, d \in \mathbb{R}$. Give brief proofs for the following inequalities.

a. $|a-b| \leq |a|+|b|$

b. $|a+b+c| \leq |a|+|b|+|c|$

c. $|a-b| \leq |a-c|+|c-d|+|d-b|$

(a)
$$
\begin{aligned}
|a-b| &= |a+(-b)| \\
&\leq |a| + |-b| \\
&= |a| + |b|
\end{aligned}
$$

(b)
$$
\begin{aligned}
|a+b+c| &= |(a+b)+c| \\
&\leq |a+b| + |c| \\
&\leq |a| + |b| + |c|
\end{aligned}
$$

(c)
$$
\begin{aligned}
|a-b| &= |(a-c)+(c-d)+(d-b)| \\
&\leq |a-c| + |c-d| + |d-b|
\end{aligned}
$$

Problem 2. Decide which of the following statements is true. Give a brief justification if the statement is valid and a counterexample if it is not.

a. Two real numbers a, b satisfy $a < b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

b. Two real numbers a, b satisfy $a \leq b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

(a) false, let $a=1$, $b=1$ (for example). Then $1 < 1+\varepsilon$ for every $\varepsilon > 0$ but $1<1$ is not valid.

(b) true

proof ($\Rightarrow$) Assume $a \leq b$ and let $\varepsilon > 0$. Then

$$a \leq b < b + \varepsilon.$$

($\Leftarrow$) Assume $a < b + \varepsilon$ for every $\varepsilon > 0$ and suppose $a > b$. Let $\varepsilon_0 = a - b$. Then

$$a < b + \varepsilon_0 = b + a - b = a,$$

which is a contradiction.

Problem 3. For each of the following statements, write its negation. Then make a guess about whether the statement itself or its negation is true.

a. For all real numbers satisfying $a < b$, there exists $n \in \mathbb{N}$ such that $a + 1/n < b$.

b. There exists a real number $x > 0$ such that $x < 1/n$ for all $n \in \mathbb{N}$.

shorthand notation for statement:

$$\exists\, x > 0,\ \forall\, n \in \mathbb{N},\ x < \frac{1}{n}.$$

negation:

$$\forall\, x > 0,\ \exists\, n \in \mathbb{N},\ x \geq \frac{1}{n}$$

(for every real number $x > 0$, there exists $n \in \mathbb{N}$ such that $x \geq \frac{1}{n}$)

Negation is true

Statement is false (there's no special positive number x that is smaller than every fraction of the form $\frac{1}{n}$).