

§ 17 Continuity

Def Let $D \subseteq \mathbb{R}$, $f: D \rightarrow \mathbb{R}$, and $a \in D$ be given.

Then f is continuous at a if $\forall \varepsilon > 0 \exists \delta > 0$
such that $|f(x) - f(a)| < \varepsilon$ whenever $x \in D$ and $0 < |x - a| < \delta$.

Further, f is continuous if it's continuous at all $a \in D$.

Remarks ① This definition is saying f is a continuous
if $\lim_{x \rightarrow a} f(x) = f(a)$.

② δ can depend on both ε and a .

Example Prove $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^2$ is continuous.

Proof We must show $f(x)$ is continuous at all $a \in \mathbb{R}$.

Let $\varepsilon > 0$ and $a \in \mathbb{R}$. Define $\delta = \min\{1, \frac{\varepsilon}{1+2|a|}\}$.

Suppose $x \in \mathbb{R}$ and $0 < |x - a| < \delta$. Then

$$\begin{aligned} |f(x) - f(a)| &= |x^2 - a^2| \\ &= |x - a||x + a| \\ &< \delta (|x + a|) \\ &\leq \delta (|x| + |a|). \end{aligned}$$

Observe that

$$|x| = |x - a + a| \leq |x - a| + |a| < \delta + |a| \leq 1 + |a|.$$

Therefore

$$\begin{aligned} |f(x) - f(a)| &< \delta (|x| + |a|) \\ &< \delta (1 + 2|a|) \\ &\leq \varepsilon \end{aligned}$$

Example Prove $f: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ given by $f(x) = \frac{1}{x}$
is continuous at 2.

Proof Let $\varepsilon > 0$. Define $\delta = \underline{\min\{1, 2\varepsilon\}}$. Suppose
 $x \neq 0$ and $0 < |x - 2| < \delta$. Then

$$\begin{aligned} |f(x) - f(2)| &= \left| \frac{1}{x} - \frac{1}{2} \right| \\ &= \left| \frac{2 - x}{2x} \right| \\ &= \frac{|x - 2|}{2|x|} \\ &< \frac{\delta}{2|x|} \end{aligned}$$

Note $2 - \delta < x < 2 + \delta$ and since $\delta \leq 1$,

$$|x| = x > 2 - \delta \geq 2 - 1 = 1$$

$$\begin{aligned} \text{Therefore } |f(x) - f(2)| &< \frac{\delta}{2|x|} \\ &\leq \frac{\delta}{2} \\ &\leq \frac{2\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Theorem Let $D \subseteq \mathbb{R}$, $f: D \rightarrow \mathbb{R}$, $a \in \mathbb{R}$

be given. Then $\lim_{x \rightarrow a} f(x) = f(a)$ if and only if

\forall sequence $(x_n) \subseteq D$ such that $x_n \neq a \ \forall n \geq 1$ and

$$\lim_{n \rightarrow \infty} x_n = a, \text{ we have } \lim_{n \rightarrow \infty} f(x_n) = f(a)$$

Proof (\Rightarrow) Let $(x_n) \subseteq D$ be a sequence such that

$x_n \neq a$ for all $n \geq 1$ and $\lim_{n \rightarrow \infty} x_n = a$. We must show

that $\lim_{n \rightarrow \infty} f(x_n) = f(a)$, which we'll do with an ϵ - N

proof. Let $\epsilon > 0$. Since $\lim_{x \rightarrow a} f(x) = f(a)$, $\exists \delta > 0$ such

that $|f(x) - f(a)| < \epsilon$ when $x \in D$ and $0 < |x - a| < \delta$.

Since $\lim_{n \rightarrow \infty} x_n = a$, $\exists N$ such that $|x_n - a| < \delta$

when $n > N$. Suppose $n > N$. Observe that

$$|f(x_n) - f(a)| < \epsilon$$

since $x_n \in D$ and $0 < |x_n - a| < \delta$ (note $|x_n - a| > 0$

since $x_n \neq a$ for all $n \geq 1$).

(\Leftarrow) Suppose by way of contradiction that $\lim_{x \rightarrow a} f(x) \neq f(a)$

That is, we suppose $\exists \varepsilon_0 > 0$ such that $\forall \delta > 0$,

$\exists x \in D$ such that $0 < |x - a| < \delta$ and $|f(x) - f(a)| \geq \varepsilon_0$.

This implies that $\forall n \geq 1$, choosing $\delta = \frac{1}{n}$, we have

$\exists x_n \in D$ such that $0 < |x_n - a| < \frac{1}{n}$ and $|f(x_n) - f(a)| \geq \varepsilon_0$.

Thus we get a sequence $(x_n) \in D$ such that

$x_n \neq a$ for all $n \geq 1$ and $\lim_{n \rightarrow \infty} x_n = a$. By assumption,

$\lim_{n \rightarrow \infty} f(x_n) = f(a)$. However, we have a contradiction

since $|f(x_n) - f(a)| \geq \varepsilon_0$ for all $n \geq 1$ implies by the

order limit theorem that $0 \geq \varepsilon_0$ but $\varepsilon_0 > 0$.

Important uses of this theorem

- ① Proving basic theorems can be done using known limit theorems for sequences
- ② Proving functional limits do not exist is easier with sequences compared to ε - δ

Example Let $f(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$.

Prove $f(x)$ is not continuous at 0.

Proof We'll show that \exists sequence $(x_n) \in \mathbb{R}$

such that $x_n \neq 0$ for all $n \geq 1$ and $\lim_{n \rightarrow \infty} x_n = 0$

but $\lim_{n \rightarrow \infty} f(x_n) \neq f(0)$. Let $x_n = \frac{1}{n}$ for each $n \geq 1$.

Then $f(x_n) = 1$ for all $n \geq 1$, and so $\lim_{n \rightarrow \infty} x_n = 0$

but $\lim_{n \rightarrow \infty} f(x_n) = 1 \neq f(0)$.

Theorem Let $f, g: D \rightarrow \mathbb{R}$ be continuous at a . Then

the following functions are also continuous at a :

- ① kf for any $k \in \mathbb{R}$
- ② $f+g$
- ③ fg
- ④ f/g as long as $g(a) \neq 0$

Proof of ① Let $(x_n) \subseteq D$ be a sequence such that

$x_n \neq a$ for all $n \geq 1$ and $\lim_{n \rightarrow \infty} x_n = a$. Since f is

continuous at a , $\lim_{n \rightarrow \infty} f(x_n) = f(a)$. Therefore, by

the Algebraic Limit Theorem, $\lim_{n \rightarrow \infty} (kf)(x_n) = k \lim_{n \rightarrow \infty} f(x_n) = kf(a)$.

Thus kf is continuous at a .

Problem 1. Give an ϵ - δ proof showing that $f(x) = x^3$ is continuous ~~at~~.

Let $a \in \mathbb{R}$, $\epsilon > 0$ and define $\delta = \min\left\{1, \frac{\epsilon}{1+3|a|+3a^2}\right\}$. Suppose $x \in \mathbb{R}$ and $0 < |x-a| < \delta$. Then

$$\begin{aligned} |f(x) - f(a)| &= |x^3 - a^3| \\ &= |x-a||x^2 + ax + a^2| \\ &< \delta |x^2 + ax + a^2| \\ &\leq \delta (|x|^2 + |a||x| + a^2) \end{aligned}$$

Observe that

$$|x| = |x-a+a| \leq |x-a| + |a| < \delta + |a| \leq 1 + |a|$$

Therefore

$$\begin{aligned} |f(x) - f(a)| &< \delta (|x|^2 + |a||x| + a^2) \\ &< \delta ((1+|a|)^2 + |a|(1+|a|) + a^2) \\ &= \delta (1 + 3|a| + 3a^2) \leq \epsilon \end{aligned}$$

Problem 2. Give an ϵ - δ proof showing that $f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ is continuous at 0.

Let $\epsilon > 0$ and define $\delta = \epsilon$. Suppose $0 < |x| < \delta$.

If $x \in \mathbb{Q}$, then $|f(x) - f(0)| = |x| < \delta = \epsilon$.

If $x \notin \mathbb{Q}$, then $|f(x) - f(0)| = 0 < \epsilon$.

Problem 3. Prove that $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ is not continuous at 0. You may assume the fact that $\sqrt{2}/n$ is irrational for all $n \geq 1$.

We'll show $\exists (x_n) \subseteq \mathbb{R}$ such that $x_n \neq 0$ for all $n \geq 1$

and $\lim_{n \rightarrow \infty} x_n = 0$ but $\lim_{n \rightarrow \infty} f(x_n) \neq f(0)$. Observe

that $x_n = \frac{\sqrt{2}}{n}$ for all $n \geq 1$ works since $\lim_{n \rightarrow \infty} \frac{\sqrt{2}}{n} = 0$

and $\lim_{n \rightarrow \infty} f(x_n) = 0 \neq f(0)$

Problem 4. Let f and g be functions that are continuous at a . Prove the following functions are continuous at a as well:

- $f+g$
- fg

Let (x_n) be a sequence such that $x_n \neq a$ for all $n \geq 1$ and $\lim_{n \rightarrow \infty} x_n = a$. Since f and g are continuous at a , $\lim_{n \rightarrow \infty} f(x_n) = f(a)$ and $\lim_{n \rightarrow \infty} g(x_n) = g(a)$.

Observe that

$$\begin{aligned} \lim_{n \rightarrow \infty} (f+g)(x_n) &= \lim_{n \rightarrow \infty} (f(x_n) + g(x_n)) \\ &= f(a) + g(a) \\ &= (f+g)(a) \end{aligned}$$

by the Algebraic Limit Theorem. Similarly,

$$\begin{aligned} \lim_{n \rightarrow \infty} (fg)(x_n) &= \lim_{n \rightarrow \infty} f(x_n)g(x_n) \\ &= f(a)g(a) \\ &= (fg)(a) \end{aligned}$$